DERIVATIVES IN THE CALCULUS

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1. Introduction. The use of derivatives, especially those of the first and second order, finds an important place in our calculus books in connection with the study of curve tracing. Such topics as extreme values, the sense of concavity of a curve, and points of inflection, are usually emphasized at an early stage of the work. These matters make relatively light demands upon the student's knowledge of limiting processes, especially if he is content to rely upon an intuitive understanding of the following proposition: *a function is increasing if its derivative is positive, and decreasing if its derivative is negative.* Without either adequate definitions or proofs, the foregoing proposition may be used to make highly plausible the usual tests, involving first and second derivatives, for extreme values, sense of concavity, and points of inflection on a curve $y=f(x)$. But most students, and, I suspect, a good many teachers, would experience considerable difficulty if they were pressed to give precise statements and proofs for the traditional theorems in this part of the calculus. There are several stumbling blocks. Just what is the meaning of the statement that a curve is concave upwards? How is a point of inflection defined? To what extent is it necessary for first and second derivatives to be continuous, or even to exist? There are some vexing questions about tangents parallel to the y-axis, and the implications about the derivative.

An elaborate treatment of these matters in elementary calculus texts would be out of place. The substance of this paper is not put forward with the suggestion that it be incorporated with a first course in calculus, but rather with the idea that instructors would welcome a rigorous presentation of the material which they are obliged to teach with a minimum of formal proofs. It is my hope that it will be of help in illuminating the possibilities of a simple and precise discussion of the problems mentioned above.

The theorems and proofs here set forth are, for the most part, not new; many of them, however, are not well known. So far as I have been able to determine, theorems 5.3 and 5.4 have not appeared elsewhere.

Throughout the paper f denotes a single-valued, real function of the real variable x. For the open interval $a<x<b$ we use the notation (a, b). The corresponding closed interval is denoted by $[a, b]$. By a neighborhood of $x=a$ is meant an open interval containing $x=a$. The statement "$f'(a)$ exists" means that the derivative exists as a finite limit. If

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = +\infty,$$

we say that f has the infinite derivative $f'(a)=+\infty$ at $x=a$. A similar definition is laid down to give meaning to the equation $f'(a)=-\infty$, When the symbols $+\infty$, $-\infty$ are used in inequalities, we observe the usual custom of writing $-\infty<x<+\infty$ for any real number x.

The law of the mean will be a fundamental tool in many parts of the paper; we shall therefore state it formally here, without proof.

1.1 (THE LAW OF THE MEAN). *Let f be defined and continuous on $[a, b]$. Let f possess a finite or infinite derivative at each point of (a, b). Then there exists a number X such that $a<X<b$ and*

$$f(b) - f(a) = (b - a)f'(X). \tag{1.11}$$

One of the important consequences of this theorem is that a function is constant throughout any interval on which its derivative is identically zero.

2. Relative extrema. Increasing functions.

DEFINITION 2.1. Let f be defined in a neighborhood of $x=a$, and suppose that, within a sufficiently small neighborhood of $x=a$, $f(x)-f(a)$ does not change sign. Then f is said to have a relative extreme at $x=a$. In particular, f is said to have a relative maximum if $f(x)\leqq f(a)$, and a relative minimum if $f(x)\geqq f(a)$.

In calculus we are usually concerned with relative extrema such that $f(x)<f(a)$ or $f(x)>f(a)$ except when $x=a$. When the situation $f(x)=f(a)$, $x\neq a$, is thus excluded, the extreme is said to be *proper*.

We shall develop tests for relative extrema, using the first derivative. We being by exploring the implications of a non-vanishing derivative.

2.11. *Let f be defined in a neighborhood of $x=a$, and suppose that f has a finite or infinite derivative at $x=a$, such that $f'(a)\neq 0$. Then, within a sufficiently small neighborhood of $x=a$, the inequalities $x_1<a<x_2$ imply the inequalities*

$$f(x_1) < f(a) < f(x_2) \quad \text{if} \quad f'(a) > 0,$$
$$f(x_1) > f(a) > f(x_2) \quad \text{if} \quad f'(a) < 0.$$

The proof follows from the observation that the difference quotient

$$\frac{f(a+h) - f(a)}{h}$$

has the same sign as $f'(a)$, provided that $|h|$ is sufficiently small.

As a consequence of theorem 2.11 we have:

2.12. *Let f be defined in a neighborhood of $x=a$, and suppose that f has a finite or infinite derivative at $x=a$. Then $f'(a)=0$ is a necessary condition for f to have a relative extreme at $x=a$.*

DEFINITION 2.2. Let f be defined on an interval, open or closed, of the x-axis. If $x_1<x_2$ implies that $f(x_1)<f(x_2)$, f is said to be increasing on the interval. If $x_1<x_2$ implies that $f(x_1)>f(x_2)$, f is said to be decreasing on the interval.

It should be noted that $-f$ is increasing if and only if f is decreasing.

It may be inferred immediately from theorem 2.11 that an increasing function cannot have a negative derivative.

2.21. *Suppose that f is defined and continuous on $[a, b]$, and that it possesses a finite or infinite derivative at each point of (a, b). Furthermore, suppose that in (a, b) $f'(x) \geqq 0$, and that $f'(x)$ does not vanish identically over any subinterval. Then f is increasing on $[a, b]$.*

It suffices to prove that $f(a) < f(b)$, for the same argument can then be applied to any subinterval. If $a < x < b$, the law of the mean, together with the hypothesis on f', implies the existence of values x_1, x_2 such that $a < x_1 < x < x_2 < b$, and

$$f(x) - f(a) = (x - a)f'(x_1) \geqq 0,$$
$$f(b) - f(x) = (b - x)f'(x_2) \geqq 0.$$

Therefore $f(a) \leqq f(x) \leqq f(b)$. Necessarily, then, $f(a) < f(b)$, for otherwise $f(x)$ would be constant, and f' would vanish identically.*

This theorem (although not necessarily in this very general form) is crucial for a number of later tests. It is unfortunate that its proof depends upon the law of the mean, for the latter theorem does not ordinarily come into a course in calculus early enough to be of use in this connection. However, students will be convinced on the basis of theorem 2.11 that f is increasing in any interval through which $f'(x) > 0$, and it may be well to omit a formal proof unless it is demanded. The next theorem, in conjunction with theorem 2.11, supplies such a proof without recourse to the law of the mean. It is not suited to the needs of a beginner, however.

2.22. *Let f be defined and continuous on (a, b), and suppose that to each point x_0 of the interval there corresponds a neighborhood within which $x_1 < x_0 < x_2$ implies that $f(x_1) < f(x_0) < f(x_2)$. Then f is increasing on (a, b).*

For lack of space, the proof of this theorem is omitted. A good student in a first course in the theory of functions should be able to construct a proof by an argument depending upon the concept of the greatest lower bound of a set of real numbers.

The next theorem furnishes a sufficient, though not necessary, condition for a proper relative extreme.

2.3 *Let f be continuous in a neighborhood of $x = a$, and suppose that f has a finite or infinite derivative at each point of the neighborhood, except possibly at $x = a$ itself. Further, suppose that within the given neighborhood $f'(x)$ is negative on one side of $x = a$, and positive on the other. Then f has a proper relative minimum or a proper relative maximum at $x = a$ according as $f'(x) < 0$ or $f'(x) > 0$ when $x < a$.*

We give the proof for the case of a minimum, using the method of contradiction. Suppose that, for some x_1 in the neighborhood, $f(x_1) \leqq f(a)$ and $x_1 \neq a$. If

* This method of proof is taken from Ch. J. de la Vallée Poussin, Cours d' Analyse Infinitésimale, Paris, 1914, I, p. 95.

necessary we can rechoose x_1, still nearer $x=a$, so that $f(x_1)<f(a)$, for f is decreasing on the left of $x=a$, and increasing on the right, by theorem 2.21 (or by 2.11 and 2.22). Now, since f is continuous, we can choose x_2 between x_1 and a in such a way that $f(x_1)<f(x_2)$. This contradicts the fact that f is decreasing on the left of $x=a$, and increasing on the right. The proof is therefore complete.

That the conditions of the theorem are not necessary for an extreme, even when f' is continuous, was pointed out long ago.*

3. Concavity. Points of inflection. In this section we shall discuss curves defined in rectangular coordinates by the equation $y=f(x)$, where f is a continuous function.

DEFINITION 3.11. Let f be defined and let its derivative exist at each point of (a, b). If the curve $y=f(x)$, throughout (a, b), lies entirely above its tangent at each point of (a, b), (the point of contact of course excepted, we say that the curve is concave upward in (a, b). The analytical requirement is that the following inequality hold when $x_1 \neq x_2$:

$$f(x_1) - f(x_2) - (x_1 - x_2)f'(x_2) > 0. \tag{3.12}$$

If the reversed inequality holds we say that the curve is concave downward. In that case the curve $y=-f(x)$ is concave upward.

This definition is closely related to the notion of a convex function, as we shall point out in detail later on, in §5.

3.21. *The curve $y=f(x)$ is concave upward on (a, b) if and only if the derivative f' exists at each point and is an increasing function on (a, b).*

We suppose first that the curve is concave upward. Then, if $x_1<x_2$, it follows from (3.12) and the inequality which results upon interchanging x_1 and x_2 that

$$f'(x_1) < \frac{f(x_2) - f(x_1)}{x_2 - x_1} < f'(x_2). \tag{3.22}$$

Hence f' is increasing.

Suppose now that f' is increasing, and that $x_1 \neq x_2$. Then, by the law of the mean,

$$f(x_1) - f(x_2) - (x_1 - x_2)f'(x_2) = (x_1 - x_2)\left[f'(\xi) - f'(x_2)\right],$$

where ξ lies between x_1 and x_2. Since f' is increasing, $f'(\xi)-f'(x_2)$ has the same sign as x_1-x_2; therefore (3.12) must hold, and the curve is concave upward.

3.3 *Suppose that f' exists and is continuous on (a, b), and that f' has a finite or infinite derivative f'' at each point of (a, b). Then the curve $y=f(x)$ is concave upward on (a, b) if and only if $f''(x) \geqq 0$ throughout (a, b) and the equality sign does not hold throughout any sub-interval.*

* E. R. Hedrick, On a function which occurs in the law of the mean, Annals of Mathematics, Series 2, vol. 7, 1906, pp. 190–192.

If the conditions are fulfilled, it follows from theorem 2.21 that f' is increasing on (a, b); the curve is then concave upward, by theorem 3.21. Conversely, if f' is increasing on (a, b), $f''(x) \geqq 0$. If the equality were to hold throughout a subinterval, the curve would be a line segment, and hence not concave upward.

Theorem 3.3 is applicable to prove that the curve $y = x^{4/3}$ is concave upward. Here $f''(0) = +\infty$. It is also applicable to the curve $y = x^4$, for which $f''(0) = 0$.

DEFINITION 3.4. If the curve $y = f(x)$ is continuous in the neighborhood of $x = a$, if it is concave upward on one side of $x = a$, and concave downward on the other, and if f has a finite or infinite derivative at $x = a$, we say that the curve has a point of inflection at $x = a$.

Before stating a test for points of inflection we observe the following very useful fact about derivatives.

3.5. *Let f be continuous in the interval $[a, a+\delta]$, where $\delta > 0$, and let the derivative f' exist in $(a, a+\delta)$. Suppose that $f'(x)$ approaches a finite or infinite limit A as $x \to a$ from the right. Then f has a finite or infinite right-hand derivative at $x = a$, given by $f'(a) = A$. A similar theorem holds for limits from the left.*

The proof is a direct consequence of the law of the mean, for if $0 < h < \delta$

$$\frac{f(a+h) - f(a)}{h} = f'(a + \theta h), \qquad 0 < \theta < 1,$$

and the result follows when we let h approach zero.

We can now state a necessary condition for a point of inflection, under certain conditions.

3.61. *Let f' exist in a neighborhood of $x = a$; let f' have a finite or infinite derivative at $x = a$, and let the curve $y = f(x)$ have a point of inflection at $x = a$. Then $f''(a) = 0$.*

It will suffice to assume that the curve is concave upward if $x < a$, and concave downward if $a < x$, since the other case is reduced to this by considering the function $-f$. Now f' is increasing if $x < a$, and it therefore approaches a finite or infinite limit as $x \to a$. This limit must be finite, and equal to $f'(a)$, by theorem 3.5. A similar argument when $a < x$ leads to the conclusion that f' is continuous at $x = a$ (and, in fact, throughout the neighborhood). It is now clear that f' has a relative maximum at $x = a$; therefore $f''(a) = 0$, by theorem 2.11.

If $f'(a) = 0$, any sufficient condition for the curve $y = f(x)$ to be concave upward in the neighborhood of $x = a$ is also a sufficient condition for a proper relative minimum of f at $x = a$. Such a condition is furnished by theorem 3.3. However, a test can be given involving the second derivative at $x = a$ only, without any implications as to concavity.

3.7. *Let f' exist in the neighborhood of $x = a$, and let $f'(a) = 0$. Suppose that $f''(a) > 0$, where $f''(a)$ may be either finite or infinite. Then f has a proper relative minimum at $x = a$.*

The proof is simple. Since

$$0 < f''(a) = \lim_{h \to 0} \frac{f'(a+h)}{h},$$

$f'(a+h)$ has the sign of h when $|h|$ is small. The result now follows by theorem 2.3. A direct proof may be given, using the law of the mean, for if $|h|$ is small and positive, $f(a+h)-f(a)=hf'(a+\theta h)$, $0<\theta<1$, and the right member of this equality is positive.

We leave it for the reader to devise a simple proof that the condition $f''(a) \geqq 0$ is necessary for a relative minimum. Some interesting examples of functions with minima not detectable by any of the usual sufficiency tests have been given by Hedrick, in the reference cited earlier in this paper.

4. Tangents. We shall discuss plane curves C of the type $y=f(x)$, where x and y are rectangular coordinates, and f is defined and continuous on some interval. If P_i is a point of C with abscissa x_i, let the inequality $P_1<P_2$ mean that $x_1<x_2$.

DEFINITION 4.11. Let P_0 be a point of C. A half line L^+ emanating from P^0 is said to be the forward tangent to C at P_0 if, given any angle of aperture less than 180°, with vertex at P_0, and L^+ within the angle, there exists a circle with center at P_0 such that if P is on C and inside this circle, and $P_0<P$, then the directed half line P_0P is also within the angle. We write $\lim P_0P=L^+$ as $P \to P_0^+$.

The backward tangent L^- at P_0 is defined in a similar manner.

It will be obvious to anyone familiar with topology that the limits L^+, L^- may be regarded as limits of functions on one topological space to another. In each case the independent variable is P, and the dependent variable is the directed half line P_0P. The limits are necessarily unique, if they exist.

DEFINITION 4.12. The curve C is said to have a tangent at P_0 if both the backward and forward tangent to C exist at P_0, and if they lie on the same straight line through P_0.

It is possible that L^+ and L^- may coincide. For example, in the case of $y=x^{2/3}$, when $P_0=0$, both L^+ and L^- coincide with the positive y-axis.

DEFINITION 4.13. If L^+ and L^- exist at P_0, and if together they form an entire straight line through P_0, this line L is called an *ordinary* tangent of C at P_0.

This definition permits us to distinguish between tangents at "ordinary" points and cuspidal tangents, such as the one mentioned in the foregoing example.

In order to discuss tangents it is convenient to use the notations $f'_+(a), f'_-(a)$ for the right-hand and left-hand derivatives of f at $x=a$. Our conventions about such statements as "$f'_+(a)$ exists, $f'_-(a)=-\infty$," etc., are the same as for the derivative f' (see §1). If $f'_+(a)$ and $f'_-(a)$ exist and are equal, their common value is the derivative $f'(a)$.

4.21. *Let f be continuous in a neighborhood of $x=a$. Then the curve $y=f(x)$ has a tangent at $x=a$ if and only if either*

(1) *$f'_-(a)$ and $f'_+(a)$ are finite and equal,*

or

(2) *$f'_-(a)$ and $f'_+(a)$ are both infinite (but not necessarily of the same sign).*

In the first case the tangent is an ordinary tangent not parallel to the y-axis. In the second case the tangent is parallel to the y-axis, but it need not be ordinary.

4.22. *Under the hypothesis of theorem* 4.21 *the curve $y=f(x)$ has an ordinary tangent at $x=a$ if and only if f has a finite or infinite derivative at $x=a$.*

The proofs of theorems 4.21 and 4.22 are left to the reader.

There are several theorems about derivatives which are important for the light they shed on the behavior of the tangent to a curve. These theorems are not new, but they are comparatively unknown. One of them, theorem 3.5, has already been mentioned. From it we conclude the following:

4.31. *Suppose that f' exists (as a finite limit) in (a, b). Then it is continuous in (a, b) except at those points where $f'(x)$ fails to approach a limit from at least one side.*

As a corollary we have:

4.32. *If f' exists and is monotonic on $[a, b]$, it is continuous there.*

It follows from this theorem that if the curve $y=f(x)$ is concave upward (or downward) on (a, b), it possesses a continuously turning ordinary tangent.

The next theorem is acribed to Darboux by de la Vallée-Poussin.*

4.33. *Let f be continuous, and possess a finite or infinite derivative at each point of $[a, b]$. Let k be a number between $f'(a)$ and $f'(b)$. Then there is a number X, $a<X<b$, such that $f'(X)=k$.*

The hypothesis implies that $f'(a)$ and $f'(b)$ are either (1) finite and distinct, (2) one finite and the other infinite, or (3) one $+\infty$ and the other $-\infty$. To prove the theorem let us first suppose that $f'(a)$ and $f'(b)$ are of opposite sign, and prove that f' vanishes at least once between a and b. Suppose, for example, that $f'(a)>0$ and $f'(b)<0$. Now f will have a maximum value on $[a, b]$, and we see from theorem 2.11 that it cannot occur at a or b. Suppose that the maximum occurs at X, where $a<X<b$. Then $f'(X)=0$, by theorem 2.12. The proof of the general case is reduced to the case just considered by defining $\phi(x)=f(x)-kx$ and observing that $\phi'(a)$ and $\phi'(b)$ are of opposite sign.

It is worth emphasizing at this point that in Darboux's theorem and the law of the mean the derivative is not assumed to be continuous, or even always finite. Many calculus texts insist upon the continuity of the derivative. One

* Ch. J. de la Vallée-Poussin, loc. cit. p. 97. For a discussion of an issue raised by this theorem see H. Lebesgue, Leçons sur l'Integration, Paris, 1928, p. 97. Lebesgue cites a paper by Darboux: Sur les fonctions discontinues, Annales de l'École Normale, 1875.

recent book even cites a function such as $f(x)=x^{2/3}$ to show that Rolle's theorem may fail when the derivative is discontinuous!

The assumption of continuity of f cannot be dropped from Darboux's theorem. This is shown by the following counter-example. Define $f(x)=x^{-1}$ if $x\neq 0$, and $f(0)=0$. Then $f'(0)=+\infty$, $f'(1)=-1$, but $f'(x)<0$ if $0<x\leqq 1$. Here is another counter-example, in which both $f'(a)$ and $f'(b)$ are finite: define $f(x)=x^{-1}$ if $x<0$, $f(x)=x^{1/2}$ if $x\geqq 0$. Then $f'(-1)=-1$, $f'(0)=+\infty$, $f'(1)=\frac{1}{2}$, but $f'(x)\neq 0$ if $-1\leqq x\leqq 1$.

Darboux's theorem affords an alternative method of proving theorem 4.32; we leave the details to the reader.

The next two theorems deal with tangents parallel to the y-axis.

4.41. *Let f be continuous in a neighborhood of $x=a$, and let the curve $y=f(x)$ have an ordinary tangent at each point of this neighborhood, except at $x=a$. At $x=a$ let there be a tangent parallel to the y-axis, but not necessarily ordinary. Then $f'(x)$ is unbounded in every neighborhood of $x=a$, but it is not necessarily true that $|f'(x)|\to\infty$ as $x\to a$.*

We apply the law of the mean to f, obtaining

$$\left|\frac{f(a+h)-f(a)}{h}\right| = |f'(a+\theta h)|, \qquad 0<\theta<1,$$

where $|h|$ may be as small as we please. The quantity on the left becomes infinite at $h\to 0$, by theorem 4.21. Therefore $f'(x)$ cannot be bounded near $x=a$. The last assertion of the theorem is provided for by the following counter-example. We define

$$f(x) = x^{1/2} + x\sin(1/x) \quad \text{if} \quad x>0,$$
$$f(0)=0, \qquad f(x) = -f(-x) \quad \text{if} \quad x<0.$$

This function is continuous, and $f'(0)=+\infty$. If $x>0$ we have

$$f'(x) = x^{-1/2}/2 + \sin(1/x) - (1/x)\cos(1/x).$$

If $x_n=2(1+4n)^{-1}/\pi$ and $u_n=(2n\pi)^{-1}$, it is found that $f'(x_n)\to+\infty$ and $f'(u_n)\to-\infty$. Therefore, when n is sufficiently large, it follows by Darboux's theorem that f' vanishes somewhere between x_n and u_n. It is then false that $|f'(x)|\to\infty$ as $x\to 0$.

4.42. *Let f be continuous in a neighborhood of $x=a$, and let the curve $y=f(x)$ have an ordinary tangent at each point of the neighborhood distinct from $x=a$. Finally, suppose that $|f'(x)|\to\infty$ as $x\to a$. Then the curve has a tangent parallel to the y-axis at $x=a$.*

The proof depends on Darboux's theorem. We may suppose the neighborhood so small that $f'(x)\neq 0$ when $x\neq a$. Then, within this neighborhood, $f'(x)$ is of constant sign when $a<x$, since otherwise it would vanish, by Darboux's theorem. The situation is similar when $x<a$. It is now clear that $f'(x)$ becomes

infinite (with definite sign) when x approaches a from one side. It follows from theorem 3.5 that both $f'_+(a)$ and $f'_-(a)$ are infinite. This completes the proof.

5. Convex functions. The meaning of concavity was discussed in §3, where a definition suitable for use in elementary calculus was introduced. We shall now consider the relation of this notion of upward concavity to the more general concept of a convex function. The latter concept is of great importance in analysis; we hope that a brief discussion of the concept will prove to be of interest in the present situation.*

DEFINITION 5.11. A function f is said to be convex on (a, b) if, for every two points on the graph of $y=f(x)$, the graph between these points lies entirely below the corresponding chord.

An analytic formulation of the condition for convexity, convenient for our purposes, may be made as follows: if $a<x_1<x<x_2<b$, then

$$\frac{f(x)-f(x_1)}{x-x_1} < \frac{f(x_2)-f(x)}{x_2-x}. \tag{5.12}$$

From the very definition it may be verified that the slope of a chord is an increasing function of the abscissa of either of its end points. Now suppose that in (5.12) we allow x_1 and x_2 to approach x. It is evident that the unilateral derivatives $f'_-(x)$ and $f'_+(x)$ must exist. Furthermore, if $x_1<x<x_2$, a little reflection will reveal that

$$\frac{f(x)-f(x_1)}{x-x_1} < f'_-(x) \leqq f'_+(x) < \frac{f(x_2)-f(x)}{x_2-x}. \tag{5.13}$$

The existence of the unilateral derivatives implies that f is continuous. The unilateral derivatives are increasing functions. For, if $x_1<x_2<x_3$, it follows from (5.13) and (5.12) that

$$f'_-(x_1) < \frac{f(x_2)-f(x_1)}{x_2-x_1} < \frac{f(x_3)-f(x_2)}{x_3-x_2} < f'_-(x_3).$$

That $f'_+(x)$ is increasing is proved similarly.

5.21. *If the function f is convex on (a, b) it is continuous there. It possesses right-hand and left-hand derivatives at each point of the interval; these derivatives are increasing functions. They are different on a set of points which is at most enumerable.*

The proof of most of this theorem has already been given. For the proof of the last assertion in the theorem we observed from (5.13) that if $x_1<x_2$

* The reader interested in the literature of convex functions may profitably consult the following references: G. H. Hardy, J. E. Littlewood, and G. Pólya, Inequalities, Cambridge, 1934, Chapter III; E. C. Titchmarsh, Theory of Functions, Oxford, 1932, pp. 172–174; O. Haupt and G. Aumann, Differential- und Integralrechnung, Berlin, 1938, I, pp. 106–112, II, pp. 57–63.

$$f'_+(x_1) < \frac{f(x_2) - f(x_1)}{x_2 - x_1} < f'_-(x_2). \tag{5.22}$$

Hence, if $x_1 < x_2$,

$$f'_-(x_1) \leqq f'_+(x_1) < f'_-(x_2).$$

Suppose that f'_- is continuous on the right at x_1. Then, if we make x_2 approach x_1 in the above inequality, we see that $f'_+(x_1) = f'_-(x_1)$. Thus the derivative f' exists at x_1. Now f'_-, being monotonic, is continuous except possibly on an enumerable set. Hence the derivative f' exists except perhaps on such a set.

5.23. *If f is such that f' exists on (a, b), the assertion that f is convex is equivalent to the assertion that the curve $y = f(x)$ is concave upward in the sense of definition* 3.11 (*or, what is equivalent, to the assertion that f' is increasing*).

It follows from theorem 5.21 that if f is convex and f' exists, it is increasing. On the other hand, if f' exists and is increasing, the inequalities (3.22) hold, and from them we conclude, if $x_1 < x < x_2$, that

$$\frac{f(x) - f(x_1)}{x - x_1} < f'(x) < \frac{f(x_2) - f(x)}{x_2 - x},$$

so that f satisfies the convexity condition (5.12).

Next we prove that a convex function is the integral of either of its unilateral derivatives.*

5.3 *If f is convex on the interval (a, b), and if c, d are points of the interval,*

$$f(d) - f(c) = \int_c^d f'_+(t)dt = \int_c^d f'_-(t)dt.$$

Suppose that $[c, d]$ is divided into n subintervals by points x_i, where $c = x_0 < x_1 < \cdots < x_n = d$. From the inequalities (5.13) and (5.22) it may be seen that if $\Delta x_i = x_i - x_{i-1}$,

$$\sum_{i=1}^{n} f'_-(x_{i-1})\Delta x_i \leqq \sum_{i=1}^{n} f'_+(x_{i-1})\Delta x_i < f(d) - f(c) = \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]$$

$$< \sum_{i=1}^{n} f'_-(x_i)\Delta x_i \leqq \sum_{i=1}^{n} f'_+(x_i)\Delta x_i.$$

Since the increasing functions f'_-, f'_+ are integrable, the proof follows from these inequalities.

Some of the important properties of a convex function were described in theorem 5.21. We now show that these properties are sufficient to characterize

* The proof indicated here was suggested by the referee. It is much simpler than the one originally included in the paper.

convexity. Let us first recall that with any function there are associated four derivates; if we write

$$Q(x, h) = \frac{f(x + h) - f(x)}{h},$$

the upper and lower derivates on the right are, respectively $D^+f(x) = \limsup_{h \to 0+} Q(x, h)$, $D_+f(x) = \liminf_{h \to 0+} Q(x, h)$, while the upper and lower derivates on the left are $D^-f(x) = \limsup_{h \to 0-} Q(x, h)$, $D_-f(x) = \liminf_{h \to 0-} Q(x, h)$. Some or all of these derivates may be infinite. It is true that $D_-f(x) \leqq D^-f(x)$; if the equality holds the common value is $f'_-(x)$. Similar remarks apply to the right-hand derivates.

5.4 *Let f be continuous on (a, b), and suppose that one of its derivates is increasing on (a, b). Then f is convex on (a, b).**

The proof depends upon a lemma which very closely resembles the law of the mean.

5.41. *If f is continuous on $[a, b]$, there exist numbers x_1, x_2 in $[a, b]$ such that*

$$Df(x_1) \leqq \frac{f(b) - f(a)}{b - a} \leqq Df(x_2),$$

where $Df(x)$ denotes any one of the derivates of f.

The assertion is obviously true if f is linear; hence we dismiss this case, and define

$$\phi(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

The function $\phi(x)$ is continuous on $[a, b]$. It vanishes at a and b, but does not vanish identically. Hence it has either a positive maximum or a negative minimum at some point X, where $a < X < b$. We shall consider the case of a positive maximum, and leave the other case to the readers. We may without loss of generality suppose that $\phi(x) > 0$ when $a < x < b$. Then it it is easily seen that for a right-hand derivate one has $D\phi(X) \leqq 0 \leqq D\phi(a)$, while for a left-hand derivate $D\phi(b) \leqq 0 \leqq D\phi(X)$. Since

$$D\phi(x) = Df(x) - \frac{f(b) - f(a)}{b - a},$$

the proof of the lemma is complete.

The proof of theorem 5.4 may now be given in three steps, as follows:

(a) The curve $y = f(x)$ contains no line segment, because one of the derivates of f is increasing, whereas the derivative of a linear function is constant.

(b) If $x_1 < x < x_2$, it follows from lemma 5.41 that there exist number x_3 and x_4 such that, for the increasing derivate $Df(x)$,

* This theorem, for the case of the particular derivate $D_+f(x)$, was proposed and proved, in a slightly different way, by my colleague, Professor Zorn.

$$\frac{f(x) - f(x_1)}{x - x_1} \leqq Df(x_3), \qquad x_1 \leqq x_3 \leqq x,$$

$$Df(x_4) \leqq \frac{f(x_2) - f(x)}{x_2 - x}, \qquad x \leqq x_4 \leqq x_2.$$

But then $x_3 \leqq x_4$, and so, by hypothesis, $Df(x_3) \leqq Df(x_4)$. Therefore

$$\frac{f(x) - f(x_1)}{x - x_1} \leqq \frac{f(x_2) - f(x)}{x_2 - x}. \tag{5.42}$$

This means that no point of an arc of the curve $y = f(x)$ lies above the corresponding chord

(c) Finally, the equality cannot hold in (5.42) at any point x between x_1 and x_2. This would mean that points P_1, P, P_2 of the curve all lie on the same straight line. By (a) and (b) there exists a point P_3 of the curve, with abscissa x_3 between x and x_2, such that P_3 lies below the chord PP_2. If P_1PP_2 were a straight line, P would lie above the chord P_1P_3, contrary to (b). This completes the proof of the theorem.

THE SIXTH ANNUAL MEETING OF THE SOUTHWESTERN SECTION

The sixth annual meeting of the Southwestern Section of the Mathematical Association of America was held at New Mexico State College of Agriculture and Mechanic Arts, State College, New Mexico, on April 28, 1942, in conjunction with the annual meeting of the Southwestern Division of the American Association for the Advancement of Science. Professor Roy MacKay,* chairman of the Section, presided over the Tuesday morning session.

The attendance was twenty-six, including the following eleven members of the Association: J. W. Branson, Eupha A. Buck, E. A. Hazlewood, W. P. Heinzman, H. D. Larsen, Roy MacKay, C. V. Newsom, Arthur Rosenthal, Nathan Schwid, P. M. Swingle, W. W. Wallis.

At the business meeting the following officers were elected for next year: Chairman, E. A. Hazlewood, Texas Technological College; Vice-Chairman, P. M. Swingle, New Mexico State College; Secretary, H. D. Larsen, University of New Mexico. It was voted to hold the 1943 meeting at a time and place selected by the officers of the Section. A vote of appreciation was extended to New Mexico State College for its generous hospitality.

The annual luncheon which followed the morning session was addressed by Professor C. V. Newsom of the University of New Mexico. His subject was, "The transition between elementary and advanced mathematics." The meeting

* Deceased, May 12, 1942.

was climaxed by an excursion on Tuesday afternoon to the White Sands National Monument.

The following papers were presented:

1. "Topology in biology, physics, and logic" by Professor P. M. Swingle, New Mexico State College.

2. "A vertical-curve sight-distance nomograph chart for use in highway construction" by Professor J. J. McKinley, New Mexico State College, introduced by the Secretary.

3. "The asymptotic expansion of the generalized hypergeometric functions" by Professor C. V. Newsom, University of New Mexico.

4. "On the synthesis of some mathematical tools" by J. A. Joseph, New Mexico State College, introduced by Professor MacKay.

5. "A note on the extraction of square roots" by Professor H. D. Larsen, University of New Mexico.

6. "A theorem involving average functions" by Dr. C. B. Barker, University of New Mexico, introduced by the Secretary.

7. "On interval-functions and associated set-functions" by Dr. Arthur Rosenthal, University of New Mexico.

8. "Characteristic functions of hypernumbers" by Professor J. B. Shaw, University of Illinois, introduced by the Secretary. (Read by title)

Abstracts of these papers follow:

1. Professor Swingle considered an abstract definition of limit, which he modified through Aristotle's four traditional types of propositions. He modified other definitions of topology in a similar manner, and sought interpretations in various sciences. He raised questions concerning the nature of abstract topologies, as well as of other topics based upon the concept of limit.

2. Professor McKinley discussed a nomograph which he constructed, and which permits the rapid estimate of the length of vertical curve required for a given grade break and sight distance, and the sight distance for a given grade break and curve length.

3. Professor Newsom recently proved a lemma which stated that certain entire functions given in the form of power series could be represented asymptotically by a sum of integrals of a comparatively simple type. He now showed that a series which includes many forms of the hypergeometric function satisfies the condition of the lemma. Consequently, a very convenient form of the asymptotic representation of such a function was obtained, valid in a large sector of the complex plane.

4. Mr. Joseph pointed out that now, more than ever before, the conservation of time and energy is a big factor in the solving of engineering problems; but that many of the standard mathematical methods are unnecessarily long and accurate for the engineer's purpose. He gave several examples showing how various problems can be solved quickly by graphical methods.

5. Professor Larsen discussed the well-known theorem: If the square root of a number consists of $2n+1$ figures, when the first $n+1$ of these have been

obtained by the ordinary method, the remaining n may be obtained by division, if the remainder arising from the division be neglected. He pointed out that the theorem is true only if the given number is a perfect square and otherwise the theorem as stated leads to a doubt in the last digit retained in the square root.

6. Dr. Barker defined a function of two variables which possessed certain desired properties relative to a given set of functions of one variable.

7. Dr. Rosenthal reported on a section of his elaboration of the second volume of H. Hahn's "Real Functions," which he is preparing on the basis of manuscripts left by Hahn. A set-function ϕ shall be called associated to a given interval function χ with respect to a given totally additive set-function, ψ, if $\chi(S_\nu)\to\phi(M)$ for the sequences $((S_\nu))$ of systems of intervals which converge to the set M in a certain sense with respect to ψ. Necessary and sufficient conditions for the existence of such an associated set-function ϕ were given. If ψ is continuous with respect to the Lebesgue measure μ_n, these conditions consist of the ψ-continuity of χ, the differentiability of χ with respect to ψ (almost everywhere), and the ψ-summability of this derivative. If ψ is not μ_n-continuous, other conditions have to be added.

8. Professor Shaw pointed out that in any algebra, associative or non-associative, there is a characteristic function (lowest degree) for each hypernumber, the terms being homogeneous in the hypernumber if we remember that the scalar coefficients have degrees. If we substitute for the hypernumber ρ_0 the hypernumber $\rho_0+mx\rho_1+m(m-1)/1\cdot 2\ x^2\rho_2+\cdots$, the resulting form can be obtained by a device commonly used in Quaternions. The substituted form is indicated by $(1+x\delta)^m\rho_0$ where $\delta\rho_0=\rho_1$, $\delta\rho_1=\rho_2$, $\cdots$. Then by tagging δ, the form F becomes $(1+x\delta_1)^m(1+x\delta_2)^m\cdots(1+x\delta_a)^m\ F(\rho_0', \rho_0'', \cdots \rho_0^{(a)})$ where δ_1 acts only on ρ_0', δ_2 on ρ_0'', etc. The operator in front may be expanded with no trouble. Professor Shaw considered the various cases arising by letting the coefficients of the powers of x vanish.

H. D. LARSEN, *Secretary*

THE TENTH ANNUAL MEETING OF THE WISCONSIN SECTION

The tenth annual meeting of the Wisconsin Section of the Mathematical Association of America was held at State Teachers College, Oshkosh, on Saturday, May 2, 1942. Miss Irene Price, chairman of the Section, presided.

There were fifty-five present, including the following nineteen members of the Association: R. H. Bardell, Ethelwynn R. Beckwith, May M. Beenken, A. C. Berry, W. W. Bigelow, M. Mirabella Boehmer, Margaret C. Eide, H. P. Evans, Fannie Hopkins, R. C. Huffer, J. F. Kenney, R. E. Langer, Morris Marden, Mary Felice, Irene Price, E. A. Nordhaus, P. L. Trump, B. R. Ullsvik, J. I. Vass.

Sessions were held in the morning and afternoon, with a noon luncheon at Hotel Raulf, and a tea, served by the mathematics department of Oshkosh State Teachers College, following the afternoon session.

The following officers were elected for the coming year: Chairman, R. H. Bardell, University of Wisconsin in Milwaukee; Secretary-Treasurer, P. L. Trump, University of Wisconsin; Program Committee, Ethelwynn R. Beckwith, Milwaukee-Downer, A. C. Berry, Lawrence College. It was voted to hold the next meeting at the University of Wisconsin in Milwaukee on May 7, 1943. Invitations to meet at Eau Claire State Teachers College and at Milwaukee-Downer were acknowledged and filed.

At the morning meeting the following papers were presented:

1. "The normal law in probability" by Professor A. C. Berry, Lawrence College.

2. "The Lemoine and Brocard points of the triangle" by Sister Mary Felice, Mount Mary College.

3. "The application of mathematics in the field artillery" by Professor W. E. Roth, University of Wisconsin in Milwaukee, introduced by Professor Bardell.

The afternoon session was devoted to a panel discussion relating to certain problems involved in the war emergency. Miss May Beenken presided, and the leaders in the discussion with their topics were:

1. Professor H. P. Evans, University of Wisconsin, "What the University of Wisconsin is doing by way of special courses to meet the national emergency."

2. Professor J. F. Duncan, Oshkosh State Teachers College, introduced by Miss Beenken, "Government sponsored courses in this emergency."

3. A. R. Luebke, Fairbanks-Morse Company, introduced by Professor Trump, "Mathematical training necessary for people entering industry."

4. Professor Glen Eye, Principal of Wisconsin High School, Madison, introduced by Professor Trump, "The place of mathematics in the high school curriculum, both for students entering college and those entering industry."

5. Professor Merlin Hayes, University of Wisconsin Extension Division, introduced by Professor Bardell, "Opportunities in the military service for college and university students."

The sessions were well attended by high school teachers, and there was a spirited discussion in the afternoon session.

Abstracts of the morning papers follow:

1. Professor Berry discussed the justification of the use of the Normal Law as an approximation simplifying the computations arising in Probability and Statistics. A simple numerical estimate for the error introduced by making this approximation was announced. This makes it possible to measure the "degree of abnormality" of a given sample whether the sample be "large" or "small." The measure applies to n-dimensional distributions whose third order moments are finite, and to the calculations of probabilities associated with convex regions. A particular and basic case treats the reliability of the Pearson chi-square test.

2. Sister Mary Felice pointed out that besides being called the minimum point, and the symmedian point of the triangle, this point has been called by German writers, Grebe's point, and by French and English writers, Lemoine's point. In neither case was there a question of priority of discovery. Grebe announced no properties of the point which had not been known before his time, but he related this point to the old problem the locus of a point the sum of the squares of whose distances from the sides of a triangle is constant. Lemoine gave an entirely new definition for the point, discovered a number of properties that had not before been noted, and recognized that it was the same point which had been before his time known in an isolated way by one or other of its properties. Some of these properties which are not usually in the ordinary text on College Geometry are of considerable interest and were briefly reviewed.

3. Professor Roth indicated that the principal reason for the failure of young men to obtain commissions in an infantry division was their inability to show proof that they had completed a course of trigonometry in high school or college. The artilleryman in particular must have a very substantial knowledge of mathematics. This knowledge is used in survey operations; determination of locations of guns and of targets on maps or improvised grids; in the use of firing tables, making corrections upon range for wind, temperature of air, barometric pressure, temperature of powder, height of the target above or below the gun, the variations in weight of projectiles, etc.; in the preparation of visibility charts, and of dead space charts; in the determination of changes in range and direction to place rounds on the target after rounds have been observed.

P. L. TRUMP, *Secretary*

ANNUAL MEETING OF THE NEBRASKA SECTION

The fourteenth regular meeting of the Nebraska Section of the Mathematical Association of America was held at the University of Omaha on Saturday, May 2, 1942. Professor J. M. Earl, chairman of the Section, presided.

The attendance was thirty-one including the following fourteen members of the Association: A. K. Bettinger, W. C. Brenke, A. R. Congdon, H. M. Cox, D. M. Dribin, J. M. Earl, W. C. Foreman, M. G. Gaba, F. S. Harper, F. E. Marrin, Lulu L. Runge, E. Marie Hove, W. T. Stratton, and A. E. White.

The following officers were elected for the coming year: Chairman, M. A. Basoco, University of Nebraska; Secretary, Lulu L. Runge, University of Nebraska; member of Executive Committee, E. M. Berry, State Teachers College, Chadron.

As Part I of the program, the following four papers were read:

1. "A test for a stabilization technique" by Professor W. A. Dwyer, Creighton University, introduced by Professor Earl.

2. "Generalized Laguerre polynomials" by Professor W. C. Brenke, University of Nebraska.

3. "Some metric properties of a vector associated with the tangent to a general curve on an analytic surface" by W. C. Foreman, Municipal University of Omaha.

4. "Pre-study examinations in mathematics" by Professor H. M. Cox, University of Nebraska.

Part II of the program consisted of a joint meeting with the Nebraska Section of the National Council of Teachers of Mathematics. The demonstrations presented were as follows:

5. "Mechanical apparatus and elementary nomograms for the construction and illustration of harmonic curves and stereographs associated with them" by L. E. Smith and A. B. Tussel of South High School, Omaha, introduced by Professor Gaba.

6. "Demonstrations by means of oscillograms of Lissajou and other curves of value in certain ultrafrequency work used by the Signal Corps" by T. T. Smith, University of Nebraska, introduced by Professor Gaba.

7. "Precision construction of the curves demonstrated above by means of the curve tracing machine" by Professor M. G. Gaba, University of Nebraska.

Part III of the program consisted of a joint luncheon with the National Council. The two papers there presented were:

8. "High lights of the convention of the National Council of Teachers of Mathematics at San Francisco" by Professor A. R. Congdon, University of Nebraska.

9. "Mathematics in the war program" by Professor W. C. Brenke, University of Nebraska.

Abstracts of some of the papers follow:

2. Professor Brenke discussed the generalization of $L_n(x)$ of Laguerre to $L_n^{(\alpha)}(x)$ of Szegö and to a still more general type $L_n^{(\alpha,\beta)}(x)$. He made a comparison of the difference and differential equations, the generating functions, and other characteristic properties.

3. Mr. Foreman used the notation of P. O. Bell in the *Bull. Am. Math. Soc.*, vol. 47. Some of the theorems demonstrated were: (1) The unique parametric net for which the directions λ and $\bar{\lambda}$ are mutually perpendicular at a point of a non-ruled analytic surface S for every choice of λ consists of the mean orthogonal curves. (2) A necessary and sufficient condition that the directions λ and $\bar{\lambda}$ be conjugate directions at a point of S for every choice of λ is that S be referred parametrically to a conjugate system of curves. (3) If the minimal curves are parametric, the R_λ-correspondent is self corresponding only along a mean orthogonal curve; and conversely.

4. In the development of a program of student guidance at the University of Nebraska, the mathematics classification examination has been integrated into an all-university measurement program. Professor Cox explained how the extended use of the examination has made necessary certain changes in the form and scope of the examination. Correlational and statistical analyses have justi-

fied the changes on a technical basis, and have, in turn, suggested instructional problems and problems of curriculum.

5. Mr. Smith and Mr. Trussell demonstrated the drawing of harmonic curves by the use of two simple machines which were equipped with cones instead of gears. These allowed the continuous passage from a given curve to others involving different frequencies. A small mirror was so connected with moving parts of the second machine that a spot of light was caused to trace a curve with such great speed that a line of light in the form of the curve appeared on the screen. Stereoscopic effects were produced by use of lights of two different colors from suitably oriented directions. The development of cylindrical curves, whose projections are Lissajou curves, was demonstrated by using transparent celluloid sheets.

6, 7. Professor Smith demonstrated, by means of oscillograms, the Lissajou and epitrochoidal curves which have been found to be of great value in certain ultrafrequency work used by the Signal Corps. All of the curves were constructed by Professor Gaba, using the University of Nebraska curve tracing machine.

LULU L. RUNGE, *Secretary*

WHAT IS A STOCHASTIC PROCESS?

J. L. DOOB, University of Illinois

1. Introduction. A stochastic process is simply a probability process; that is, any process in nature whose evolution we can analyze successfully in terms of probability. We shall not attempt an exhaustive description. On the empirical side, a discussion of the nature of probability would take us too far afield (and might sidetrack us into philosophy), and on the mathematical side the definitions would require too much high-powered mathematics. We shall limit ourselves to a description of a stochastic process in simple terms, followed by a discussion of a few important particular types. A stochastic process was described above as an empirical entity. On the mathematical side, certain concepts have characteristics in close correspondence with those of these processes, and will be called mathematical stochastic processes. The point is that although at an elementary level probability courses frequently deal with urns, dice and events, a strictly mathematical treatment is possible, with no immediate empirical flavor.

There is a popular prejudice that probability is the subject which deals with gambling games, and perhaps with life insurance and statistics, but is otherwise useless. Moreover its principal method is the calculation of permutations and combinations, than which nothing could be more boring. This calculation looms large in elementary courses, however, only because calculations need little mathematical preparation, either for student or teacher. It is considered easier

to perform long calculations than to develop a general point of view, bolstered by theory. This may be good pedagogy, up to a certain point, but gives students a false picture of the subject. The basic reason for the applicability of permutation-combination theory to the study of probability will be noted below. It will be seen, however, that considerably more than the counting of favorable and unfavorable events by such methods is necessary to deal with the problems to be discussed.

2. Concept of a chance variable. Perhaps the basic difference between the older mathematical probability and that of the last fifteen years lies in the stress now put on the concept of a chance variable, and in the development of the whole subject in terms of that concept. In this spirit, we shall develop the idea of a stochastic process from that of a chance variable.

A chance variable, sometimes called a stochastic variable, or a random variable, or (especially by statisticians) a variate, is (non-mathematically-speaking) the numerical result of an experiment to which probability analysis is to be applied. There are various possible results, with varying probabilities assigned to them. Thus let N_t be the number of telephone subscribers who will initiate a call tomorrow in Chicago between 4 P.M. and t minutes after 4. Then, fixing t, the practical determination of the necessary number of central telephone facilities needed to prevent more than a given percentage of lost calls is based on assumptions about N_t. For each t, certain probabilities suggested by theoretical considerations are assigned to the various possible N_t values $0, 1, \cdots$. The character of the chance variable N_t depends on t. For example, the more probable values of N_t increase as t does. In many cases the chance variable y under consideration can take on any one of a continuous set of values (for example let y be the value of t when the first subscriber to initiate a call after 4 does so). In such cases individual values of y may each have 0 probability, and intervals of y-values are assigned probabilities. The probability characteristics of any chance variable y are determined by the specification of every probability of the type. Prob. $\{y<k\}$, where k is any number. Thence the probability that y satisfy other types of condition, for example Prob. $\{a \leqq y \leqq b\}$, is determined. The function

$$F(k) = \text{Prob. } \{y < k\}$$

is called the distribution function of y.

Now suppose $y_1, \cdots, y_n$ are n chance variables. To discuss them together, not only the probability relations of each y_j must be given (that is, its distribution function), but we need also the combined distribution: If $k_1, \cdots, k_n$ are any numbers, the probability $F(k_1, \cdots, k_n)$ that $y_j<k_j$ simultaneously for all j is supposed known. All the probability relations of $y_1, \cdots, y_n$ are determined by the knowledge of the "distribution function" F.

Any set of values of n chance variables $y_1, \cdots, y_n$ determines a point P in n-dimensional space. A condition imposed on the y_j can be interpreted as a condition that P lie in some portion of this space. If the probability distribution

of $y_1, \cdots, y_n$ is determined by setting the probability that P lie in a given region R proportional to the integral of e raised to a second degree polynomial, integrated over R, then $y_1, \cdots, y_n$ are said to be normally distributed, or to have a Gaussian (n-variate) distribution. The normal distributions are the most common distributions of theoretical statistics because they really do arise frequently in practice, and also because their common statistical parameters are easy to compute.

3. Concept of a stochastic process. Having finished all these preliminaries, we finally come to the concept of a stochastic process. As we have said, a stochastic process in practice is any process whose evolution we find it possible to follow and predict in terms of probability. Now the evolution of such a process is measured by a number (or can be expressed in terms of such measurements) registered in some way, perhaps continuously (at each instant of time), perhaps at discrete times (say every hour), depending on the nature of the process and our analysis of it. Each measurement is by hypothesis a chance variable. From this point of view, a stochastic process is,—and this is the usual definition,—merely a family of chance variables $\{y_t\}$ where t represents the time. Here t may run through all numbers, or only through the integers, or more generally through any set of numbers. If the process consists of a recurring experimental procedure, it is frequently convenient to think of the y_t with t (an integer) ranging to ∞ in both directions even though physically we cannot make (and cannot have made) these infinitely many measurements.

Throughout this exposition, we have not introduced any pure mathematical analysis. For example our "definition" of a chance variable was merely a description of a certain type of experimental procedure, and not a very explicit one at that. This is not the place to go into the proper mathematical definitions. We can only say that the mathematical chance variable is a certain type of function (a measurable function), so that a mathematical stochastic process is a family of (measurable) functions. The analytical study of probabilities is the study of measurable functions. Some of the simple examples we shall discuss below are far simpler to describe physically than to analyze mathematically. What makes an elementary treatment of probability possible is that in many problems the chance variables concerned can only assume a finite number of values, and the time parameter t is allowed to run through only finitely many values. Because of these two characteristics, such problems can always be solved by combinatorial methods. Any finite repetition of a gambling game has these simple properties; and this is one reason for the central place of gambling games in elementary treatments of probability. In our discussion here we are considering the stochastic processes of nature, and carefully evading a proper mathematical analysis (which incidentally dates back less than ten years).

4. Examples. We conclude with several examples of stochastic processes. First consider repeated coin tossing. Here the family of chance variables is $y_1, y_2, \cdots$, with $y_j=1$ (heads) or $y_j=0$ (tails), with respective probabilities

p and $1-p$. The chance variable y_j corresponds to the jth toss. If the coin is evenly balanced, $p=1/2$. It is supposed as usual that the probability of a set of n tosses containing m heads and $n-m$ tails in some given order is $p^m(1-p)^{n-m}$. It is convenient in the mathematical analysis, as we have remarked above, to think of the tosses as continuing in both directions in time, so that we have $\cdots, y_0, y_1, \cdots$. Physically this is impossible, conceptually it is easy to swallow (but even if one gags, the results are easy to translate in terms of finitely many y_j), and the mathematical basis is not very difficult. The translation just noted is of course essential if the theorems are to make any empirical sense at all. A more important interpretation of the same mathematical setup would be the succession of male and female births in a community, or simple examples of Mendel's laws of heredity.

Of this stochastic process we note first that there is no change in the probability relations with time; that is, the probability relations of any $y_m, \cdots, y_n$ are the same as those of $y_{m+1}, \cdots, y_{n+1}$. Such a process is called stationary, or temporally homogeneous. Now what are proper questions that can be asked of such a process? There are various well known ones. For example, the number of heads in n throws divided by n approaches p, the probability of heads in a single throw: formally, if according to the usual rules the probability that

$$\lim_{n\to\infty} \frac{y_1 + \cdots + y_n}{n} = p$$

is calculated, the answer is 1. Although the tosses are independent, there is thus still an average character, a fact which has caused a considerable amount of head shaking, and induced peculiar remarks in well known books. Knowing that $s_n/n \to p$ $(s_n = y_1 + \cdots + y_n)$ one might ask how quickly $\sigma_n = s_n/n - p$ goes to 0, or whether the probability distribution of the chance variables $\sigma_1, \sigma_2, \cdots$ has some limiting form, as n increases. These questions have been answered by the law of iterated logarithm, which states that $\lim_{n\to\infty} \sup \sqrt{n}\,\sigma_n/\sqrt{2pq \log\log n} = 1$ with probability 1, and the central limit theorem, which states that $\sqrt{n}\,\sigma_n$ has more and more nearly a normal distribution as $n \to \infty$.

In the above example, the y_j were mutually independent. The next most simple type of connection is that of a Markoff process. Consider the chance variable $s_1, s_2, \cdots$ in the previous example. Evidently if $n > m$, knowing $s_1, \cdots, s_m$; that is, knowing the number of heads obtained in each toss up through the mth, considerably restricts s_n, the number of heads obtained in n tosses. For example, $s_n \geqq s_m$. It is easy to compute the probability of the various values of s_n if $s_1, \cdots, s_m$ have preassigned values. The answers will, of course, depend on these preassigned values. Now actually it is evident, since $s_n = s_m + y_{m+1} + \cdots + y_n$, where $y_{m+1}, \cdots, y_n$ are independent of $s_1, \cdots, s_m$, that the assigned value of s_m is all that is relevant to the computation. The conditional probabilities depend only on the value assigned to s_m, not on those assigned to $s_1, \cdots, s_{m-1}$. Such a process is called a Markoff process; formally the hypothesis is that whenever $t_1 < t_2$, the probability distribution of y_{t_2} calculated

under the assumption that y_t has been assigned values for all $t \leqq t_1$ depends only on the value assigned to y_{t_1}.

Let y_t be the size of a population under statistical analysis. Whenever the population's change depends on the state of the population at a given moment, but, this being known, is independent of how the present state was attained, the y_t determine a Markoff process. It is then evident that Markoff processes will have general application in statistical studies of the population trends of a species and in epidemiology (if the number of infected individuals depends only on those already infected). Important questions to be asked here are: what is the asymptotic character of y_t for large t, is there a limiting value, any kind of periodicity, etc.?

Let y_t be a chance variable depending on the time t, which we assume runs through all values. Suppose that the increments of y_t: $(y_{t+h}-y_t)$ over non-overlapping time intervals are independent, that is if $t_1 < \cdots < t_n$ we suppose that $y_{t_2}-y_{t_1}, \cdots, y_{t_n}-y_{t_{n-1}}$ are mutually independent. The stochastic process is then called a process with independent increments, or a differential process. These arise in many connections. For example let $y_t = N_t$ where N_t was defined above in the telephone example. Then the y_t determine a differential process: the number of subscribers initating a phone call in any time interval can under ordinary circumstances be supposed independent of the calls offered before this time interval (or later). Or, to take an example from physics, let y_t be the number of radioactive disintegrations of a given substance by time t. Again we have a differential process. Or let y_t be the amount of money to be paid out by an insurance company to its claimants between time 0 and time t. Again the process can frequently be assumed a differential process.

In the examples of differential processes considered above, it is clear that y_t, considered as a function of t has as graph a set of horizontal lines, no matter how the process turns out. The proof that in general the y_t of a differential stochastic process are continuous in t except for non-oscillatory discontinuities (jumps)* is quite complicated, and is less than ten years old. Aside from simple changes of scale, there is only one differential process in which the functions y_t are actually continuous in t (with probability 1)—that in which the increments $y_{t+h}-y_t$ have normal distributions. The best known application of this process is to the haphazard Brownian movements of small particles immersed in a liquid. About 35 years ago it was shown that if y_t is a coördinate value of such a particle at time t, the movement can be analyzed fairly accurately by considering the y_t as the determining family of chance variables of a differential stochastic process whose increments have normal distributions. Thus the probability analysis predicts continuous curves as the particle trajectories. That is certainly desirable, but it was noted at once that this same probability analysis predicts infinite

* The meaning of this statement is the following. The probability can be computed that y_t as a function of t will be continuous at a point, everywhere continuous, etc. The probability that the y_t of a differential process will have any non-oscillatory discontinuity is 0 (if very minor restrictions are imposed on the process to eliminate degenerate cases).

velocities for the particles, and that the length of a trajectory between any two of its points is infinite. This may seem remarkable physically, but man can predict anything after it happens, and in fact physicists have asserted that the particle movements as observed seem to have these peculiar properties.

We go finally to a simple but very important type of stationary stochastic process. Suppose that the distribution of y_t is normal, and even that for any $t_1, \cdots, t_n$ $y_{t_1}, \cdots, y_{t_n}$ have an n-variate normal distribution (for all n). We shall call such a process a stationary normal process. It is easy to see that apart from a scaling factor, and a centering constant, such a process is completely determined by a single function $\rho(h)$, the correlation coefficient of the two chance variables y_t, y_{t+h}, which measures the connection between these chance variables. (This is independent of t since the process is stationary.) Necessary and sufficient conditions are known that a function be such a correlation function, but many questions are still unanswered about these processes, and many more will remain unformulated until the need arises. The simplest case is the degenerate case $\rho(h) \equiv 1$, when y_t is independent of t: the process never changes from its initial state, and the theory of probability is quite superfluous. At the other extreme is the case $\rho(h) = 0$ for all $t \neq 0$, when y_s, y_t are independent of each other if $s \neq t$. The first non-trivial class of stationary normal stochastic processes to investigate, is certainly the subclass of Markoff processes, of which the above two cases are degenerate special cases. For a Markoff process, aside from these two cases, the connection between y_t and y_{t+h} goes down exponentially, as $h \to \infty : \rho(h) = e^{-c|h|}$ (c a positive constant). In work on the Brownian movement dating back about nine years, and giving a better approximation than the earlier work, it was found that the particles had finite velocities after all (but infinite accelerations) and in fact that each component of the velocity of a particle at time t can be considered as a chance variable y_t, where the y_t determine one of the Markoff processes just described.

In this case of a Markoff process, with $\rho(h) = e^{-c|h|}$, it is known that the y_t are continuous functions of t, with probability 1, but it is not known generally which correlation functions $\rho(h)$ determine processes with continuous y_t.

Another example of a stationary normal process arises in electricity. The spontaneous thermal movements of the electrons in any wire cause current fluctuations in all electric circuits. If y_t is the current in a given wire at time t, y_t determines a stationary normal stochastic process. The correlation function depends on the particular circuit, and on the resistances, capacities, and inductances. This phenomenon is important in radio since the current fluctuations cause receiver noises which cannot be entirely eliminated. These current fluctuations are a disturbing influence in making any kind of delicate electrical measurements. A somewhat similar phenomenon occurs in radio tubes, known as the shot effect. The study (not yet completed) of these electrical disturbances is thus the study of certain types of stochastic processes.

GELFOND'S SOLUTION OF HILBERT'S SEVENTH PROBLEM

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1. In 1900 D. Hilbert presented his famous list of twenty-three problems which he regarded as the outstanding questions awaiting solution by mathematicians of the future. [1] The seventh problem is concerned with irrational and transcendental numbers and in particular with the following question. *If $\omega \neq 0$, 1 is algebraic and θ is algebraic but not rational, is ω^θ transcendental or at least not rational?* As specific examples he mentioned $2^{\sqrt{2}}$ and $e^\pi = i^{-2i}$.

The first contribution to this problem was given by the Russian mathematician A. Gelfond [2] in 1929. He proved that e^π is actually transcendental and indicated how his method could be used to prove transcendentality whenever θ belongs to an imaginary quadratic field. The extension to real quadratic fields was given by C. L. Siegel (unpublished) and R. A. Kuzmin [3] in 1930.

Gelfond returned to the question later and in 1934 he could give a complete solution [4] of Hilbert's problem which was followed in 1935 by a more elementary solution obtained independently by Th. Schneider [5], one of Siegel's pupils. The proof of Gelfond, though more advanced, has quite a simple basis and gives a beautiful example of teamwork between algebraical and analytical ideas. For this reason it deserves to be much better known. The original paper is not easily accessible and is full of disturbing misprints, especially in the French text. Though there is an excellent review by K. Mahler in *Zentralblatt für Mathematik*, vol. 9, pp. 53–54, a detailed exposition of the ideas of the proof would still seem to be of some value to the mathematical community. The present version is based upon an analysis presented to a seminar at Stanford University in November 1941 and is a fairly free account.

2. We start by recalling some algebraical concepts. An *algebraic number* α of degree m over the rational field $\mathbf{R}$ is the root of an algebraic equation of degree m,

$$A_0 x^m + A_1 x^{m-1} + \cdots + A_m = 0, \tag{2.1}$$

with rational integral coefficients, irreducible over $\mathbf{R}$. The other roots are known as the *conjugates* of α. $N(\alpha) = A_m/A_0$ is the *norm* of α. If $A_0 = 1$, α is called an *algebraic integer*. We can always find a rational integer A such that $A\alpha$ is an algebraic integer. It suffices to take $A = A_0$. If the roots of (2.1) are $\alpha_1 = \alpha$, $\alpha_2, \cdots, \alpha_m$, we denote the largest of the numbers $|\alpha_j|$ by $\mu(\alpha)$.

The number α generates an *algebraic number field* $\mathbf{K}(\alpha)$ of degree m over $\mathbf{R}$ consisting of all polynomials in α of degree $\leqq m-1$ with coefficients in $\mathbf{R}$. Every element γ of $\mathbf{K}(\alpha)$ is an algebraic number of degree d over $\mathbf{R}$, where d is a divisor of m, and the conjugates of γ are obtained by replacing α by its conjugates in the defining polynomial. If β in an algebraic number of degree n over $\mathbf{K}(\alpha)$, *i.e.*, the root of an equation of degree n with coefficients in $\mathbf{K}(\alpha)$ and irreducible over $\mathbf{K}(\alpha)$, then the *extension field* $\mathbf{K}(\alpha, \beta)$ consists of all polynomials in β of degree $\leqq n-1$ with coefficients in $\mathbf{K}(\alpha)$ and the elements are of degree $\leqq mn$ over $\mathbf{R}$. Similarly for further extensions. The following property of algebraic numbers is

fundamental in all work involving transcendentality.

LEMMA 1. *If γ lies in an algebraic field of degree m over R and if A is a rational integer such that $A\gamma$ is an algebraic integer, then either $\gamma=0$ or*

$$|\gamma| \geqq A^{-d}[\mu(\gamma)]^{1-d} \geqq A^{-m}[\mu(\gamma)]^{1-m}. \tag{2.2}$$

The Lemma merely expresses the fact that $|N(A\gamma)| \geqq 1$.

3. In addition to the algebraic notions we shall need two results of analytical nature. The first is a less common form of Kronecker's theorem on Diophantine approximations [6] and the second is the theorem of Jensen [7].

LEMMA 2. *Given a matrix (a_{ij}) with m rows and n columns, $n>2m$, where the a_{ij} are complex numbers and $|a_{ij}| \leqq A$. Let P be a given positive number. Then there exist n rational integers $N_1, N_2, \cdots, N_n$, such that*

$$\left| \sum_{i=1}^{n} N_i a_{ij} \right| \leqq 1/P, \qquad j = 1, 2, \cdots, m, \tag{1}$$

$$\sum_{i=1}^{n} |N_i| \geqq 1, \tag{2}$$

$$|N_i| \leqq [2^{3/2} nAP]^{2m/(n-2m)}, \qquad i = 1, 2, \cdots, n. \tag{3}$$

For the proof, let $\nu_1, \nu_2, \cdots, \nu_n$, be a set of nonnegative integers each less than or equal to an integer B which will be disposed of later. There are $(B+1)^n$ possible distinct choices of such sets. Put

$$\sum_{i=1}^{n} \nu_i a_{ij} = a_j = c_j + id_j, \qquad i = 1, 2, \cdots, m.$$

In a space of $2m$ dimensions we mark the points

$$(c_1, d_1, c_2, d_2, \cdots, c_m, d_m).$$

These points are located in a $2m$-dimensional cube of side $\leqq 2nAB$. If we lay off the length $2^{-1/2}P^{-1}$ along the side, we get at most $2^{3/2}nABP+1$ intervals which in turn determine a subdivision of the big cube into at most $(2^{3/2}nABP+1)^{2m}$ subcells, cubes or parallellepipeda, no side of which can exceed $2^{-1/2}P^{-1}$. If now B is so chosen that

$$(B+1)^n > (2^{3/2}nABP+1)^{2m},$$

there will be at least one of the $2m$-dimensional subcells which contains two points of our set, P_1 and P_2 say. Suppose $(\nu_{11}, \nu_{12}, \cdots, \nu_{1n})$ and $(\nu_{21}, \nu_{22}, \cdots, \nu_{2n})$ are the corresponding integers. We put $N_i=\nu_{1i}-\nu_{2i}$, $i=1, 2, \cdots, n$. There is obviously at least one $N_i \neq 0$. The points P_1 and P_2 being in the same subcell, their coördinates in homologous position differ by at most $2^{-1/2}P^{-1}$. A simple computation shows that (1) holds, (2) has already been verified. Now $(B+1)^n$ is certainly greater than $(C+1)^{2m}$ if $B^n \geqq C^{2m}$. This gives in our case the value of B occurring in (3).

LEMMA 3. *Let $G(z)$ be holomorphic in $|z| \leqq R$ and have the zeros $z_1, z_2, \cdots, z_n$ in $|z| < R$. Then*

$$|G(0)| \leqq R^{-n} |z_1 z_2 \cdots z_n| \max_{0 \leqq \theta < 2\pi} |G(Re^{i\theta})|.$$

The inequality is trivial if $G(0) = 0$, so we can assume $G(0) \neq 0$. For each zero z_j we form the function

$$B_j(z) = \frac{R(z - z_j)}{R^2 - z\bar{z}_j},$$

which vanishes at $z = z_j$, has a simple pole at $z = R^2/(\bar{z}_j)$ outside the circle $|z| = R$, and is of absolute value one on the circle. Put

$$F(z) = G(z)\left[\prod_{j=1}^{n} B_j(z)\right]^{-1}.$$

This function is holomorphic in $|z| \leqq R$ and

$$\max |F(z)| = \max |G(z)| \quad \text{on} \quad |z| = R.$$

But

$$F(0) = \frac{1}{2\pi i}\int_{|t|=R} \frac{F(t)}{t}\, dt,$$

so that

$$|F(0)| \leqq \max |F(z)| = \max |G(z)|, \qquad |z| = R,$$

and

$$G(0) = (-R)^{-n}\left[\prod_{j=1}^{n} z_j\right] F(0).$$

Combining, we get the desired inequality. It is obvious, and it was observed already by Jensen, that the proof does not call for a complete knowledge of the zeros of $G(z)$ in $|z| < R$, but that the more zeros we can locate, the smaller will $|G(0)|$ have to be. This observation will be used repeatedly below.

4. We are now ready for the proof of Gelfond's theorem which he formulated as follows:

If α and β, different from zero and one, are algebraic while $\eta = (\log \alpha)/(\log \beta)$ is irrational, then η is transcendental.

The theorem asserts that no irrational algebraic power of an algebraic number can be algebraic, with obvious and trivial exceptions. This is obviously equivalent to Hilbert's problem.

Let us begin by giving a brief outline of underlying ideas of the proof. By assumption, α and β are algebraic and determine an algebraic field $\mathbf{K}(\alpha, \beta)$. Suppose now that η is also algebraic. We can then form the extension field $\mathbf{K}(\alpha, \beta, \eta)$.

We then form an auxiliary function $f(x)$, having values in $\mathbf{K}(\alpha, \beta)$ for every rational integral value of x. The simplest such function would be a polynomial in x with coefficients in $\mathbf{K}(\alpha, \beta)$, but such a function would not enable us to bring in η by any obvious process. Instead, Gelfond uses the old stand-by, the exponential function. He forms an exponential polynomial,

$$f(x) = \sum_{k=-q}^{q} \sum_{l=-q}^{q} C_{kl}\alpha^{kx}\beta^{lx}, \tag{4.1}$$

where the C_{kl} are rational integers and q is a positive integer, which will be disposed of later. Here $\alpha^{kx} = \exp[kx \log \alpha]$ with some definite but arbitrary definition of the logarithm, and similarly for β^{lx}. This function obviously has values in $\mathbf{K}(\alpha, \beta)$ when x is a rational integer. But more can be asserted. We have

$$f_s(x) \equiv f^{(s)}(x)(\log \beta)^{-s} = \sum_{k=-q}^{q} \sum_{l=-q}^{q} C_{kl}(k\eta + l)^s\alpha^{kx}\beta^{lx}, \tag{4.2}$$

so that $f_s(x)$ has values in $\mathbf{K}(\alpha, \beta, \eta)$ for every integral x and $s = 0, 1, 2, \cdots$.

The function $f(x)$ contains $(2q+1)^2$ parameters, the C_{kl}, which together with the value of q are at our disposal. In particular, we can take q as large as we please. By virtue of Lemma 2 the C_{kl} can be determined in such a manner that a fairly large set of the algebraic numbers $f_s(j)$ are very small in absolute value, so small in fact that condition (2.2) is violated and they, therefore, have to be zero. In other words, we can impose a certain number of zeros on $f(x)$ at prescribed integers while keeping the coefficients as integers of limited size. Gelfond distributes the zeros between several points, but it is simpler to place them all at the origin. This accumulation of zeros makes itself felt in a fairsized neighborhood of the origin and Jensen's theorem shows that $|f(x)|$ has to be quite small, say for $|x| < q^{2/3}$. By Cauchy's formula for the derivatives, this extends also to $|f^{(s)}(x)|$ for values of s not too large. But this in turn makes a still larger set of the algebraic numbers $f_s(j)$ so small that they have to be zero by Lemma 1. Thus we get a large number of additional zeros of large multiplicity in a neighborhood of the origin and by Jensen's theorem this forces $|f(x)|$ to be still smaller in a still larger neighborhood. By alternate use of Lemmas 1 and 3 we could show that $f(x)$ and all its derivatives vanish at all integers which of course implies $f(x) \equiv 0$. Actually, two applications of Jensen's theorem give us enough conditional equations on the C_{kl} to conclude that they are either all zero or η is a rational number. The first possibility is excluded by the construction of $f(x)$ which ensures $\sum\sum |C_{kl}| \geqq 1$, while the second violates the hypothesis of the theorem. We conclude that η cannot be algebraic.

5. The crux of the proof is right at the start in the application of Lemma 2, in particular the choice of m and P. We shall try to elucidate this point somewhat. It is desired to choose m, P, and integers C_{kl} with $\sum\sum |C_{kl}| \geqq 1$, such that

$$|f_s(0)| \equiv \left| \sum_{k=-q}^{q} \sum_{l=-q}^{q} C_{kl}(k\eta + l)^s \right| \leqq 1/P, \qquad s = 0, 1, \cdots, m-1, \tag{5.1}$$

while condition (2.2) is violated. Lemmas 1 and 2 assert that this can be done provided certain inequalities hold.

Suppose that η is algebraic of degree h over $\mathbf{R}$, and that $E\eta$ is an algebraic integer, E being a rational integer. Then $E^s f_s(0)$ is also an algebraic integer of degree $\leqq h$ over $\mathbf{R}$ if the C_{kl} are rational integers. If $|C_{kl}| \leqq B$, we have

$$\mu[f_s(0)] \leqq B(2q+1)^2\{q[1+\mu(\eta)]\}^s.$$

In order to violate (2.2) it is thus sufficient that

$$P > E^{sh}\{B(2q+1)^2 q^s[1+\mu(\eta)]^s\}^{h-1}, \tag{5.2}$$

and this is certainly satisfied for large m if*

$$P > \{B(E_1 q)^m\}^h, \tag{5.3}$$

where E_1 is a fixed integer exceeding $2E[1+\mu(\eta)]$.

In the other direction we have to take into account the conditions imposed by Lemma 2. First we have $n=(2q+1)^2$, $m<2n$, $a_{k,l;s}=(k\eta+l)^s$, $-q\leqq k$, $l\leqq q$, $s=0, 1, \cdots, m-1$, so we can take $A=\{q[1+\mu(\eta)]\}^{m-1}$. It is clear that P must be much larger than B if (5.3) is to hold. This requires $m=o(q^2)$, but on the other hand it is advantageous to have m as large as possible. Condition (3) of Lemma 2 will be satisfied if

$$B > \{P(E_1 q)^m\}^{(m/q)^2} \tag{5.4}$$

where E_1 may be chosen as the same integer as in (5.3).

These inequalities still give considerable leeway for the quantities involved. They show that P must be larger than both B and q^m and that $B>q^{(m/q)^2}$, but they do not show which should be the larger, B or q^m. The later application of Jensen's theorem requires, however, that q^m is dominant.

We can satisfy all the requirements by Gelfond's choice,†

$$B = 3^{q^2}, \qquad m = \left[q^2 \frac{\log\log q}{\log q}\right], \qquad P = \exp\left\{\gamma q^2 \frac{\log q}{\log\log q}\right\}, \tag{5.5}$$

where γ depends only upon the field $\mathbf{K}(\alpha, \beta, \eta)$. It is of course supposed that q is a large number.

We know now that it is possible to choose integers C_{kl}, not all zero, such that $|C_{kl}| \leqq 3^{q^2}$, and $x=0$ is a zero of the function $f(x)$ defined by (4.1), the multiplicity of the zero being at least m. Further

$$|f(x)| \leqq \sum\sum |C_{kl}||\alpha^{kx}\beta^{lx}| < 3^{q^2}(2q+1)^2 e^{\delta q|x|}$$

or

$$|f(x)| \leqq \exp[2q^2 + \delta q|x|], \qquad q \geqq 2 \tag{5.6}$$

where δ depends only upon α and β.

* It is enough that $m \geqq h-1$ and $2q \geqq \mu(\eta)$. The estimates required here and below are, up to a certain point, extremely crude.

† $[u]$ is the largest integer $\leqq u$.

6. We come now to the first application of Lemma 3. Let us put $G(z)=f(a+z)$ where a is any point on the circle $|x|=q^{2/3}$ (any power of q less than the first would do). Consider a circle with center at $z=0$ and radius q and substitute in Lemma 3. $G(z)$ is known to have a zero of multiplicity m at $z=-a$. Hence

$$\begin{aligned}|G(0)| = |f(a)| &\leqq \left(\frac{q^{2/3}}{q}\right)^m \exp\left[2q^2 + \delta q(q+q^{2/3})\right] \\ &= q^{-m/3} \exp\left[(2+\delta)q^2 + \delta q^{5/3}\right] \\ &< \exp\left[-\tfrac{1}{3}q^2 \log\log q + 2(1+\delta)q^2\right].\end{aligned}$$

Hence by the principle of the maximum*

$$|f(x)| < \exp\left[-\tfrac{1}{6}q^2 \log\log q\right], \qquad |x| \leqq q^{2/3}, \tag{6.1}$$

if q is sufficiently large.†

We now apply Cauchy's formula

$$f^{(s)}(x) = \frac{s!}{2\pi i}\int \frac{f(t)dt}{(t-x)^{s+1}}$$

to the circle $|t|=q^{2/3}$ and choose $|x| \leqq \frac{1}{2}q^{2/3}$. Replacing $s!$ by s^s, we get

$$|f^{(s)}(x)| < 2(2sq^{-2/3})^s \exp\left[-\tfrac{1}{6}q^2 \log\log q\right]$$

and the estimate

$$|f^{(s)}(x)| < \exp\left[-\tfrac{1}{12}q^2 \log\log q\right], \quad |x| \leqq \tfrac{1}{2}q^{2/3}, \quad 0 \leqq s \leqq q^2/(\log q) \tag{6.2}$$

for all large q. Since $f_s(j)=f^{(s)}(j)\,(\log\beta)^{-s}$, this implies

$$|f_s(j)| < \exp\left[-\tfrac{1}{24}q^2 \log\log q\right] \tag{6.3}$$

for $j=0\pm 1, \pm 2, \cdots, \pm[\frac{1}{2}q^{2/3}]$, $s=0, 1, 2, \cdots, [q^2/(\log q)]$, provided q is sufficiently large.

We show next that (6.3) implies that $f_s(j)=0$ by virtue of Lemma 1. We have

$$f_s(j) = \sum_{k=-q}^{q}\sum_{l=-q}^{q} C_{kl}(k\eta+l)^s \alpha^{kj}\beta^{lj}.$$

Choosing rational integers A, B, C, D, and E such that $A\alpha$, $B\beta$, C/α, D/β, and $E\eta$ are algebraic integers, we see that $(ABCD)^{jq}E^s f_s(j)$ is an integer of the field

* The maximum of the absolute value of an analytic function is reached on the boundary of the domain under consideration.

† The integer q is subject to several conditions which assign lower bounds for it. Such bounds involving η where noted in connection with formula (5.3). The choice of m and P in (5.5) requires $\log\log q>0$ or $q>e$. (6.1) calls for $\log\log q \geqq 12(1+\delta)$, a bound depending only upon α and β. (6.2) requires a lower bound which is an absolute constant while (6.3) gives a bound depending upon β. In (6.4) we again have a lower bound for q depending upon δ, but a closer investigation shows that this does not impose any new restrictions on q. (7.1) imposes the last restriction which again depends upon β.

$\mathbf{K}(\alpha, \beta, \eta)$. Suppose that the degree of the latter is H which is at most equal to the product of the degrees of α, β, and η over $\mathbf{R}$. For the maximum of the conjugates of $f_s(j)$ we have the estimate

$$\mu[f_s(j)] \leqq 3^{q^2}(2q+1)^2\{q[1+\mu(\eta)]\}^s\tau^{2jq},$$

where $\tau = \max[\mu(\alpha), \mu(\beta), \mu(1/\alpha), \mu(1/\beta)]$. Hence if (2.2) is to hold we must have

$$|f_s(j)| \geqq [(ABCD)^{jq}E^s]^{-H}\{\mu[f_s(j)]\}^{1-H} > \exp[-\sigma q^2],$$

where σ is a fixed constant depending only upon the field, but not upon q. This contradicts (6.3), however, for large q and, therefore, we must have $f_s(j)=0$.

Thus $f(x)$ has a zero at each integer j with $|j| \leqq \frac{1}{2}q^{2/3}$ and every zero has a multiplicity of at least $[q^2/(\log q)]$. We now resort to Jensen's theorem once more. We put $G(z)=f(a+z)$, $|a|=q^{4/3}$, $R=q^{3/2}$. Every known zero within the circle $|z|=R$ satisfies the inequality $|z_\nu| < 2q^{4/3}$. Hence using (5.6)

$$|f(a)| < \exp\left\{\frac{q^{8/3}}{\log q}\log(2q^{-1/6}) + 2(1+\delta)q^{5/2}\right\}$$

so that

$$|f(x)| < \exp[-\tfrac{1}{12}q^{8/3}], \qquad |x| \leqq q^{4/3}, \tag{6.4}$$

for all large q.

It is clear that this process can be continued indefinitely and would grind out more and more zeros of $f(x)$ at the rational integers and would thus ultimately lead to the conclusion that $f(x) \equiv 0$.

7. For our purposes it is enough to observe, however, that using Cauchy's formula for $f^{(s)}(0)$ together with (6.4) we obtain

$$|f^{(s)}(0)(\log\beta)^{-s}| < \exp\{-\tfrac{1}{24}q^{8/3}\}, \qquad 0 \leqq s \leqq q^{5/2}. \tag{7.1}$$

We compare this estimate with formulas (5.1) and (5.2). Taking $1/P$ equal to the right hand side of (7.1), we see that (5.2) is satisfied, *i.e.*, (2.2) is violated and, therefore, $f^{(s)}(0)=0$ for $s \leqq q^{5/2}$.

But this gives a system of $[q^{5/2}]$ linear equations

$$\sum_{k=-q}^{q}\sum_{l=-q}^{q}(k\eta+l)^s C_{kl} = 0 \tag{7.2}$$

to be satisfied by the $(2q+1)^2$ unknowns C_{kl}. Since the latter are known to be not all zero, any $(2q+1)^2$-rowed determinant must be zero. In particular,

$$\det|(k\eta+l)^s| = 0, \qquad -q \leqq k, l \leqq q,\ 0 \leqq s \leqq 4q(q+1). \tag{7.3}$$

But this is a Vandermonde determinant which vanishes if and only if two columns are equal. This implies that η is rational against our assumptions. This completes the proof of Gelfond's theorem.

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THE INSTANTANEOUS MOTION OF A RIGID BODY

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1. Combination of instantaneous rigid motions. The assertion that a rigid body is rotating about the x-axis with a certain angular velocity, and rotating at the same time with another angular velocity about the y-axis, puts a strain on the imagination of a student meeting this form of statement for the first time. He is relieved to find that the statement is not one that needs to be interpreted literally, being merely a somewhat irresponsible substitute for a clear formulation in mathematical terms. What is meant is merely that if the body is regarded as a continuous distribution of matter, the vector velocity of each of its points is the geometric resultant of the velocity which *would be* associated with that point by the first rotation and the one which it would have in the second rotation.

The analysis of the most general instantaneous motion of a rigid body can be carried through in terms of ideas of corresponding simplicity. (The phrase "instantaneous motion" is understood for the purposes of this paper to be concerned throughout with the velocities of the points considered, not with their accelerations, which would present a more complex problem.) The notions involved are of course essentially vectorial. In particular, the theory offers notable concrete or semi-concrete exemplification of the significance of the distributive law for vector multiplication. The main features of the theory are presented below from this point of view.* The presentation lays no claim to novelty; its purpose is merely to give one possible arrangement of the details in order for consecutive reading.

* For a formulation in quite different language see for example R. S. Ball, The Theory of Screws, Dublin, 1876, pp. xix–xxiv; for further comparison see W. F. Osgood, Mechanics, New York, 1937, Chapter V.

The problem is that of characterizing the motion itself, without reference to the forces which produce or control it. The characterization is independent also of the size and shape of the body considered. If its motion or that of any three-dimensional portion of it is specified, the velocity which a particle at any point of space would possess if rigidly attached to it is determined. It is to be supposed for the purposes of the present study that *each point of space* has a definite vector velocity assigned to it, in a manner consistent with the *condition of rigidity* presently to be laid down. Any such distribution of velocities, or velocity field, will be called for brevity a *rigid motion*. For similar purposes of abbreviation any set of velocities which there is occasion to consider will be called a *motion*, whether subjected explicitly to the hypothesis of rigidity or not.

It will be seen that in a sense to be carefully defined *the most general rigid motion is either a translation or a rotation or the resultant of a translation and a rotation.*

The *condition of rigidity* is that *for every pair of points P_1, P_2, the vector velocity of P_1 and the vector velocity of P_2 have equal components along the line P_1P_2.* This expresses for instantaneous motion the property that the distance between any two points of the body remains invariable. As between the two opposite directions along the line, it is naturally to be understood that the equal components agree in direction as well as in magnitude.

The discussion here, as already remarked, is concerned exclusively with the motion itself, not with the conditions by which it is produced. It is of no consequence whether any material body to which the conclusions may be applied is *capable* of deformation or not, provided that the velocities which its particles actually possess under specified circumstances are such that the condition of rigidity is fulfilled.

If any two "motions," *i.e.* sets of velocities defined for the points of space, are denoted by M' and M'', their *resultant*, represented symbolically by $M'+M''$, is the motion in which the velocity of each point is the resultant or vector sum of the velocities assigned to that point by M' and M'' separately. It is an immediate consequence of the definitions that *the resultant of any two rigid motions is a rigid motion*, since for each pair of points P_1, P_2 the components of the resultant velocities along P_1P_2 are obtained by algebraic addition of components which are separately equal for the two points.

2. Simple rigid motions: translation and rotation. A *translation* is a motion in which all points have equal vector velocities; that is to say, in less technical but colloquially* more descriptive language, the velocities of all points are equal and parallel. It follows from the definition that a translation is a rigid motion.

Another fundamental type of motion, called a *rotation*, can be described as follows:

* In an endeavor to minimize technicality of expression, the word *velocity* will be used interchangeably for the vector velocity and for its magnitude, when no misunderstanding seems possible. If the reader desires to have the distinction appear in the record he can of course accomplish this with brevity by using the word *speed* on occasion for the magnitude of the vector velocity.

a) There is a straight line, called the *axis* of the rotation, all of whose points have zero velocity.

b) The velocity of any point not on the axis is perpendicular to the plane containing the point and the axis.

c) All points at equal distances from the axis have equal velocities (the word "equal" being used again as an abbreviation for "equal in magnitude").

d) Points at different distances from the axis have velocities proportional to those distances.

e) All velocities correspond to the same "sense" or direction of turning about the axis.

It will be seen presently that the content of all this verbal description can be condensed into a simple vector formula.

A rotation is "obviously" a rigid motion, in the sense that intuition recognizes it as a kind of motion possible for a rigid body. It is another matter to show formally that it satisfies the definition of a rigid motion in terms of the velocities of an arbitrary pair of points. To give such a proof by the methods of elementary geometry would be a somewhat substantial exercise. It can be done in a few lines by means of vector algebra, for the reason that the geometric relations involved are precisely those which vector algebra recognizes as fundamental. Another reason for stressing this proof is that it appears in some way to form the backbone of the entire theory; simple as it is, all the other proofs to be given subsequently are so much simpler as to be scarcely more than a succession of "remarks."

By the specifications describing a rotation above, the ratio of the velocity of an arbitrary point P to its distance from the axis is the same for all points of space. This ratio is the *angular velocity* of the rotation. If the same unit of length is used for measuring velocity and for measuring distance, the angular velocity is measured in radians per unit of time. All the essential characteristics identifying a particular rotation are conveniently represented by a vector ω lying in the axis, with magnitude numerically equal to the angular velocity just defined, in the adopted scale of measurement, and pointing in the direction in which the rotation would carry a right-handed screw. This ω is called the *vector velocity of rotation*. Like a vector representing a force in the dynamics of a rigid body, it is to be thought of as lying in a definite line, but may be laid off from any point of that line as initial point.

Let O be an arbitrary point of the axis, and let ρ be the vector from O to an arbitrary point P of space. By a check of magnitude and direction it is seen at once that *the velocity of P is represented by the vector product* $\omega \times \rho$. For any specified ω this product defines a set of vectors throughout space having the characteristics of a rotation.

A rotation being given with ω as its vector representation, and a fixed point of reference O on its axis, let ρ_1 and ρ_2 be the vectors from O to a pair of arbitrary points P_1, P_2 anywhere in space. The vector velocities of P_1 and P_2 are $\omega \times \rho_1$ and $\omega \times \rho_2$. Their components in the direction from P_1 toward P_2, if the distance

P_1P_2 is denoted by D, are $D^{-1}\omega \times \rho_1 \cdot (\rho_2 - \rho_1)$ and $D^{-1}\omega \times \rho_2 \cdot (\rho_2 - \rho_1)$, and D times the difference between these components is

$$(\omega \times \rho_2 - \omega \times \rho_1) \cdot (\rho_2 - \rho_1) = \omega \times (\rho_2 - \rho_1) \cdot (\rho_2 - \rho_1) = 0,$$

because of the distributive law for the scalar product, the distributive law for the vector product, and the fact that $\omega \times (\rho_2 - \rho_1)$ is perpendicular to $\rho_2 - \rho_1$ (or, as an alternative formulation for the last step, the fact that a scalar triple product is zero if two of its factors are alike). The vanishing of the difference means that the condition of rigidity is fulfilled.

As a trivial special case, the assignment of zero velocities to all points will be regarded alternatively as defining a translation of zero magnitude or a rotation of zero magnitude about an arbitrary axis, or will be referred to simply as a zero motion.

3. Analysis of general rigid motion. A description of the most general rigid motion is now obtained through the following sequence of observations.

I. *A rigid motion in which three non-collinear points have zero velocities is a zero motion.* Let O_1, O_2, O_3 be three such points. Let P be any point outside their plane. By the condition of rigidity P can not have any velocity component along any of the lines O_1P, O_2P, O_3P. That is to say, all three lines must be perpendicular to the velocity of P, *if any*. Since they do not lie in one plane, this is impossible, and the words "if any" indicate a condition contrary to fact; the velocity of P must be zero. As for points in the plane $O_1O_2O_3$, let Q be any such point, and let O_4 be a point outside the plane. By the preceding proof the velocity of O_4 is zero, and repetition of the argument with O_4 in place of one of the three points originally given shows that Q has zero velocity.

II. *A rigid motion in which two distinct points have zero velocities is a rotation about the line of these points as axis.* Let O_1, O_2 be the given points. Let P_0 be a point outside their line. If P_0 has zero velocity, all points have zero velocity by the preceding paragraph, and the motion can be regarded as a zero rotation. If P_0 has a velocity, this velocity can have no component along O_1P_0 or O_2P_0, by the condition of rigidity, and so must be perpendicular to both lines and to the plane of the three points. Let v_0 be the magnitude of the velocity of P_0, and r the perpendicular distance of P_0 from the line O_1O_2. Let M denote the given rigid motion; of the two opposite rotations about O_1O_2 with angular velocity v_0/r, let R denote the one that gives to P_0 the velocity which it possesses in M, and $-R$ the other. Then the resultant of M and $-R$, for brevity $M-R$, is a rigid motion in which O_1, O_2, and P_0 have zero velocities, and so all points have zero velocities, by reference to the preceding paragraph once more. That is to say, M as a set of velocities for the points of space is identical with R.

III. *A rigid motion in which a point O has zero velocity is a rotation about an axis through O.* The assertion that if there is a point at rest there must be a whole line of points at rest recalls the similarly striking fact in solid geometry that if two planes have a point in common they must have a whole line in common. It will be seen that one fact is a consequence of the other. In the formulation

of the proof, trivial specializations which have the effect merely of bringing back the conditions of the preceding paragraphs will not be explicitly enumerated.

Let P_1 be a point distinct from O, and ϕ_1 its vector velocity. Since ϕ_1 must be perpendicular to OP_1, by the condition of rigidity, the plane through P_1 perpendicular to ϕ_1 contains O. Let p_1 denote this plane. Let P_1' be any point of p_1 outside the line OP_1. The velocity of P_1' must be perpendicular to OP_1' and to P_1P_1', and so perpendicular to p_1, since P_1 has no component of velocity along P_1P_1' and O has no velocity at all. A selected reference point in p_1 outside OP_1 may then be used to extend the conclusion to points of this line, in analogy with the final step in the proof of I. *All* points of p_1 have their velocities, if any, perpendicular to p_1.

Let P_2 be a point outside p_1, ϕ_2 its velocity, and p_2 the plane through P_2 perpendicular to ϕ_2. By reasoning similar to that just presented, p_2 passes through O, and the velocities of all points of p_2 are perpendicular to p_2.

Since p_1 and p_2 have the point O in common, they intersect in a line. If O_1 is another point of the line of intersection its velocity, if any, must be perpendicular to both planes. As this is impossible, O_1 has zero velocity, and reference to II completes the proof.

IV. *If O is an arbitrarily chosen point, the most general rigid motion is resultant of a rotation about an axis through O and a suitable translation.* (It is understood naturally that either the rotation or the translation may in particular be zero.) Let M be the given rigid motion, let ϕ be the vector velocity of O, and let T be the translation in which all points have this vector velocity, while $-T$ is the opposite translation, with velocity $-\phi$. Then $M-T$, interpreted as $M+(-T)$, is a rigid motion in which O has no velocity, and so by III is a rotation about an axis through O. If this rotation is denoted by R, M is the resultant of R and T.

It is to be noted that only a single point of the axis, not the whole axis, is arbitrary.

V. *The resultant of a rotation and a translation perpendicular to the axis of the rotation is a rotation of equal angular velocity about a parallel axis.* Let R denote the rotation, with ω as the vector representation of its angular velocity, and T the translation, with velocity ϕ; the hypothesis requires that $\omega \cdot \phi = 0$.

To demonstrate formally a fact which is obvious to geometric intuition, namely that there is a line of points to which R assigns the velocity $-\phi$, let $\psi = \omega \times \phi/\omega^2$, where ω^2 denotes the square of the magnitude of ω, let O be an arbitrary point of the axis of R, and let O' be the corresponding point such that the vector OO' is ψ. Then the velocity $\omega \times \psi$ which R gives to O' is in fact $-\phi$, as may be seen either by application of the rule for evaluating a vector triple product or by elementary interpretation of the successive operations of simple vector multiplication.

As O describes the axis of R, the point O' describes a parallel line, and the rigid motion $R+T$, giving zero velocity to all points of this line, is a rotation R' about it. The equality of the angular velocities is recognized by comparing

the vector velocities of a pair of corresponding points O' and O in the respective rotations, the line joining them being perpendicular to both axes.

VI. *The most general rigid motion is resultant of a rotation and a translation parallel to the axis of the rotation.** On the basis of this analysis the motion is called a *screw motion.* Let O be an arbitrary point, and let the given motion M be expressed by IV as a rotation R about an axis through O plus a translation T. Let T be resolved into component translations T_0 and T_1, parallel and perpendicular to the axis of R. By V, R and T_1 can be combined into a rotation R_0 about a parallel axis, and M is then the sum of R_0 and T_0.

With the conclusions IV and VI the theory attains a certain stage of completeness. Some additional facts are nevertheless deserving of emphasis.

Of these perhaps the most striking relates to the combination of rotations about intersecting axes. It follows at once from III that the resultant of two such rotations is a rotation about an axis through the point of intersection, since this point has zero velocity in the resultant motion. More specifically, let O be the intersection, let ω_1 and ω_2 be the vectors representing the given rotations, let P be an arbitrary point of space, and let ρ be the vector OP. The resultant of the velocities given to P by the two rotations separately is

$$\omega_1 \times \rho + \omega_2 \times \rho = (\omega_1 + \omega_2) \times \rho,$$

which is the same as the velocity corresponding to a single rotation represented by the vector $\omega_1+\omega_2$. *Instantaneous rotations about intersecting axes can be added vectorially, as an immediate consequence of the distributive law for vector multiplication.*

This fact is the basis for the resolution of an instantaneous rotation into component rotations about a set of coordinate axes.

In the combination of a rotation with a *non-vanishing* translation parallel to its axis, the velocity given to any point by the rotation, having no component parallel to the axis, can not cancel the velocity due to the translation; there is no point with zero velocity, and the resultant motion is not equivalent to any single rotation alone.

Suppose a given rigid motion is resolved in any way into a rotation R_1 and a translation T_1, and again into a rotation R_2 and a translation T_2. Then one may write the equations.

$$R_1 + T_1 = R_2 + T_2, \qquad R_2 = R_1 + T_1 - T_2.$$

(Such manipulation does not involve the setting up of any new type of algebra; it is merely a symbolism for representing comprehensively the corresponding elementary combinations of the vector velocities for the various points of space.) Let the translation $T_1 - T_2$ be resolved into components T', T'' perpendicular and parallel to the axis of R_1. By V, the resultant of the rotation R_1 and the

* Discovery of this theorem is ascribed to G. Mozzi, 1763; see Encyklopädie der mathematischen Wissenschaften, vol. 4: 1, article IV 2, H. E. Timerding, Geometrische Grundlegung der Mechanik eines starren Körpers, pp. 125–189; p. 143.

translation T' is a rotation R' with equal angular velocity about a parallel axis. In the equation

$$R_2 = R' + T''$$

it follows from the preceding paragraph that T'' must vanish. *The rotations R_1 and R_2 have equal angular velocities about parallel axes; the translations T_1 and T_2 have equal components in the common direction of these axes.* To restate a part of this conclusion in different words, *a given rigid motion has a vectorial angular velocity ω which is determined in magnitude and direction by the rigid motion itself, and is independent of any choice of a particular point of reference as origin.*

If in particular it is supposed that neither T_1 nor T_2 has any transverse component, T' vanishes, R_2 is the same as R_1, and T_2 is the same as T_1; *the resolution given by* VI *is uniquely determined.*

4. Supplementary notes. By way of additional comment, attention may be directed to certain facts with regard to translations which were not needed in the main body of the discussion. The obvious fact that a translation satisfies the condition of rigidity was noted at an early stage. It is almost as easy to see that a rigid motion in which all the velocities are parallel is necessarily a translation. For if P_1, P_2 are any two points such that the line joining them is not perpendicular to the common direction of the velocities, equality of the components along P_1P_2 implies that the total velocities of P_1 and P_2 are equal; from an equation $v_1 \cos\theta = v_2 \cos\theta$ it follows that $v_1 = v_2$, if $\cos\theta \neq 0$. If the line P_1P_2 is perpendicular to the direction of translation, the velocities of P_1 and P_2 are equal to that of any third point P_3 outside the plane through P_1 and P_2 perpendicular to that direction, and so again equal to each other.

Less obvious perhaps at the outset, but easily recognized when the general theorems have been established, is the proposition that if the velocities in a rigid motion are all equal they must be parallel, and the motion is a translation once more. For if the motion is resultant of a translation and a rotation in which the latter is not zero, the magnitudes of the velocities are certainly not all equal.

In summary, of the properties of rigidity, parallelism, and equality of magnitude, any two imply the third.

The reader may be interested to show as an "exercise" that *the most general rigid motion is either a rotation or* (in an infinite variety of ways) *resultant of two rotations about non-intersecting axes.*

A more extensive exercise, which is of importance in itself and will serve to throw much light in retrospect on the theory that has been developed, is to carry through a corresponding discussion in two dimensions, that is to say, with consideration only of points in one plane, and with the assumption that all velocities lie in that plane. A noteworthy difference between two dimensions and three is found in connection with the theorem numbered VI: *Every plane rigid motion is either a translation or a rotation.*

A PAIR OF GENERATORS FOR THE SIMPLE GROUP $LF(3,3)$

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The linear fractional group $LF(3, 3)$ is simple and of order 5616. We wish to prove the

THEOREM. *The group $LF(3, 3)$ can be generated by two elements of orders 13 and 8, respectively.*

The group $LF(3, 3)$ is simply isomorphic with the collineation group G of the finite plane projective geometry $PG(2, 3)$. If $P_0, P_1, \cdots, P_{12}$, are the thirteen points of the $PG(2, 3)$, the group G can be exhibited as a permutation group of degree 13 on the letters $P_0, P_1, \cdots, P_{12}$.*

For the sake of brevity, we shall designate a point P_i by its subscript i. The $PG(2, 3)$ is then given by the array

0	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12	0
3	4	5	6	7	8	9	10	11	12	0	1	2
9	10	11	12	0	1	2	3	4	5	6	7	8

The thirteen lines $l_0, l_1, \cdots, l_{12}$ of the $PG(2, 3)$ are the thirteen columns of the array reading from left to right. An element of the group G is then a permutation on the subscripts of the P's which has the property that it sends lines into lines. We shall prove the theorem by showing that every collineation of the $PG(2, 3)$ can be written as a product of the collineations

$$a = (0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12),$$
$$b = (1\ 5\ 11\ 8\ 3\ 4\ 10\ 12)(2\ 9\ 7\ 6),$$

of orders 13 and 8, respectively. We can verify by direct trial that a and b are indeed collineations.

We need for the sequel the collineations

$$c = a^{11}ba^{12} = (1\ 6\ 2\ 5\ 9\ 11\ 12\ 4)(3\ 10\ 8\ 7),$$
$$d = a^5ba^2 = (1\ 4\ 9\ 7\ 3\ 5)(2\ 8)(6\ 10\ 11),$$
$$e = aba^8 = (1\ 4\ 6)(2\ 12\ 8)(3\ 5\ 10)(7\ 11\ 9),$$
$$f = a^4ba^3 = (2\ 5\ 10\ 8\ 4\ 6)(3\ 9)(7\ 11\ 12),$$
$$g = a^{10}ba = (1\ 9\ 3)(4\ 6\ 5\ 10\ 7\ 11)(8\ 12).$$

The collineations

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* For the necessary definitions, etc., see L. E. Dickson, Linear Groups; O. Veblen and W. H. Bussey, Finite projective geometries, Trans. A. M. S., vol. 7, pp. 241–259, April 1906; R. D. Carmichael, Introduction to the Theory of Groups of Finite Order.

(1) $$c^2,\ b^4,\ b^5,\ b,\ c,\ d^3,\ b^3,\ c^4,\ b^6,\ b^2,\ b^7,$$

leave the point 0 invariant and send the point 1 into the points

$$2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ 12,$$

respectively. The inverses of the collineations (1) leave point 0 invariant and send the points 2, 3, $\cdots$, 12, respectively, into the point 1. Hence there is a collineation

$$A_{ij}, \qquad (i \neq j;\ i, j = 1, \cdots, 12)$$

which leaves point 0 invariant and which sends an arbitrary point i ($\neq 0$) into an arbitrary point j ($\neq 0, i$). The collineations A_{ij}, for all i's and j's, are products of the collineations a and b.

The collineations

(2) $$f^4,\ ed^5f^2g^3,\ ed^5f^2g^3f^4,\ (ed^5f^2g^3f^4)^2,\ ed^5f^2g^3f^2,\ f^2,\ (ed^5f^2g^3)^2,\ ed^5,$$

leave the line $l_0 \equiv 0, 1, 3, 9$, pointwise invariant and send the point 2 into the points

$$4,\ 5,\ 6,\ 7,\ 8,\ 10,\ 11,\ 12,$$

respectively. Hence, as before, there is a collineation

$$B_{ij}, \qquad (i \neq j;\ i, j = 2, 4, 5, 6, 7, 8, 10, 11, 12)$$

which leaves the line l_0 pointwise invariant and sends an arbitrary point i ($\neq 0, 1, 3, 9$) into an arbitrary point j ($\neq 0, 1, 3, 9, i$). The collineations B_{ij}, for all i's and j's, are products of the collineations a and b.

The collineations

(3)
$$\begin{aligned}
C_2 &= g^3 = (4\ 10)(6\ 7)(5\ 11)(8\ 12),\\
C_4 &= g^3f^2 = (2\ 10)(5\ 7)(6\ 12)(8\ 11),\\
C_5 &= ed^5f^2 = (2\ 11)(4\ 7)(6\ 8)(10\ 12),\\
C_6 &= ed^5f^4 = (2\ 7)(4\ 12)(5\ 8)(10\ 11),\\
C_7 &= ed^5f^2g^3ed^5 = (2\ 6)(4\ 5)(8\ 10)(11\ 12),\\
C_8 &= ed^5 = (2\ 12)(4\ 11)(5\ 6)(7\ 10),\\
C_{10} &= f^2g^3 = (2\ 4)(5\ 12)(6\ 11)(7\ 8),\\
C_{11} &= ed^5f^2g^3f^4ed^5 = (2\ 5)(4\ 8)(6\ 10)(7\ 12),\\
C_{12} &= ed^5f^2g^3f^2ed^5 = (2\ 8)(4\ 6)(5\ 10)(7\ 11),
\end{aligned}$$

leave the line l_0 pointwise invariant and also leave invariant the points

$$2,\ 4,\ 5,\ 6,\ 7,\ 8,\ 10,\ 11,\ 12,$$

respectively. Furthermore, as we can see from its form, the collineation C_i, for a given i, actually alters all points other than 0, 1, 3, 9, and i. Each of these nine collineations is a product of the collineations a and b.

Now, let w, x, y, z, be an arbitrary set of four independent (that is, no three collinear) points of the $PG(2, 3)$ and let W, X, Y, Z, be another such set of points. Since a collineation of the $PG(2, 3)$ is necessarily a projective collineation, there is just one collineation K which sends w, x, y, z into W, X, Y, Z, respectively. We must show that K can be expressed as a product of the collineations a and b.

The collineation

$$K_1 = a^{W-w}$$

sends the point w into the point W. (The integers $0, 1, \cdots, 12$ now play a double rôle; they are, on the one hand, names of the points and subscripts on the collineations of types A, B, and C. As such, sums and differences of these integers should be reduced modulo 13 to the range $0, 1, \cdots, 12$. On the other hand, the thirteen integers will occur as exponents on the collineations. As such they can, of course, be reduced modulo the order of the collineation.) Let the collineation K_1 send the points x, y, z into the points x', y', z', respectively. The collineation

$$K_2 = a^{-W}A_{x'-W,X-W}a^W$$

leaves W invariant, sends x' into X, and sends y' and z' into y'' and z'', say, respectively. Let W and X be on line l_m; the collineation

$$K_3 = a^{-m}B_{y''-m,Y-m}a^m$$

leaves W and X invariant, sends y'' into Y, and z'' into z''', say.

The points W, X, Y are not collinear; let W and X be on line l_r, X and Y on line l_s, W and Y on line l_t. No one of the lines l_r, l_s or l_t will contain either z''' or Z. Consider the three collineations

$$k = a^{-r}C_{Y-r}a^r,$$

$$k' = a^{-s}C_{W-s}a^s,$$

$$k'' = a^{-t}C_{X-t}a^t,$$

Each of these collineations leaves the points W, X, Y invariant and hence leaves each of the lines l_r, l_s, l_t invariant. However, k leaves the line l_r pointwise invariant but not the other two, k' leaves l_s pointwise invariant but not the other two, and k'' leaves l_t pointwise invariant but not the other two. Hence k, k', and k'' are distinct collineations. Since each has the form of a collineation of type C and there are just four points in the $PG(2, 3)$ not on l_r, l_s, l_t, one (and only one) of the collineations k, k', k'' sends z''' into Z. Let K_4 be this collineation.

The collineation

$$K = K_1K_2K_3K_4$$

sends the points w, x, y, z into the points W, X, Y, Z, respectively and is a product of the collineations a and b. The theorem follows.

DISCUSSIONS AND NOTES

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The department of Discussions and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

CAN A ROBOT CALCULATE THE TABLE OF LOGARITHMS?

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I think it can, by the following method:* Let r_1 denote the square root of 10, and let subsequent r's be determined by

$$r_{n+1} = \sqrt{r_n}.$$

Table I shows the r's to eight significant figures together with their common logarithms. Eight digits are used because they just fill up the keyboard of the usual calculator, and anyone using it as a teaching device can obtain six or seven accurate digits and thus show how the ordinary five place table is rounded off. *Any number between* 1 *and* 10 *can be expressed as the product of selected* r*'s, and no* r *need be used more than once.* This will be obvious after reading the example given below. And so the logarithm of such a number is merely the sum of the logarithms of the selected r's.

Example. Find $\log 235.7 = 2 + \log 2.357$.

First find the largest r in Table I which is not greater than 2.357 (it is r_2), and divide it into this number. The quotient is 1.3254385. Next divide this quotient by the largest r that is not greater than the quotient (it is r_4), and this next quotient is 1.1477824. Next we divide this last quotient by the largest r that is not greater than the quotient (it is r_5), and we obtain 1.0680942.

Now at each step we divide the quotient of the preceding step by the largest r that is not greater than the quotient. *No quotient can be as great as the* r *that was used to obtain it*; for if it were, we could have divided by the preceding r. Twelve more divisions are required to reach the point where the last quotient, to eight significant figures, is unity. We thus show that 2.357 is equal to the product of r_2, r_4, r_5, r_6, r_7, r_8, r_{10}, r_{12}, r_{15}, r_{17}, r_{18}, r_{19}, r_{20}, r_{22}, r_{24}. Hence our mantissa is the sum of the logarithms of these r's, and we have

$$\log 235.7 = 2.37235957.$$

This may sound like a long process, but it takes only about ten minutes on an old fashion hand-crank machine. It is not necessary to write down the successive quotients, but it is necessary to keep up with which r's were used.

* This method is similar to Briggs' original method. Encyclopaedia Britannica, eleventh edition, article on "Logarithm," Vol. 16, p. 875.

THEOREM. *The error in the logarithm of any number, calculated by this method, will not exceed in numerical value the quantity* $16 \cdot k \cdot 10^{-k}$, *where k is the number of significant figures used throughout the calculation.*

To prove this we note that in performing any one division the error in the quotient could not exceed one unit in the kth significant figure. Hence, if we let $p(x)$ denote the number of divisions needed to calculate $\log x$, the error in the last quotient could not exceed $p(x) \cdot 10^{-k+1}$. We note further that if n is large (greater than 20), any small increment in r_n, multiplied by $.434 \cdots$, will approximate the corresponding increment of $\log r_n$. Consequently the error in $\log x$ due to accumulated errors of division cannot be greater than $4.34 \cdot p(x) \cdot 10^{-k}$. To this must be added the error that might accumulate from adding up $p(x)$ values of $\log r$, and this could not exceed $.5 \cdot p(x) \cdot 10^{-k}$. *Thus the total error in* $\log x$ *cannot exceed numerically the value* $4.84 \cdot p(x) \cdot 10^{-k}$.

Now the greatest value that $p(x)$ could have is not more than the largest value of n in the table, since no r is used more than once. Let N denote this largest value of n, and N will be an integer which approximately satisfies the equation

$$2^{-N} = 10^{-k}.$$

Hence N is approximately equal to $3.32 \cdot k$. Substituting this value for $p(x)$ in the formula for maximum error in $\log x$ gives the desired result.

It might be pointed out that usually $p(x)$ is much smaller than N, so that in an isolated calculation, it would be better to use say $5 \cdot p(x) \cdot 10^{-k}$ as the upper limit of error instead of the larger value $16 \cdot k \cdot 10^{-k}$ which is a uniform upper limit of error for all such calculations.

Example 1. In calculating $\log 235.7$, we note that $p(2.357)$ equals 15. Hence the error in our result cannot exceed $5 \cdot 15 \cdot 10^{-8}$ or .00000075.

Example 2. It is desired to calculate a 20-place table of logarithms by this method. How many significant figures should be used? Twenty-five digits would suffice since the error will not exceed $16 \cdot 25 \cdot 10^{-25}$ or $400 \cdot 10^{-25}$.

The Robot, since we are building it only in our imagination, might as well be a deluxe affair. It will consist of a dial A, to carry the number whose logarithm is being calculated, a machine B to do division, a machine C to total up the logarithm, and a complicated governor. Dial A carries six digits, and B and C twenty-five digits each; for we are to make out a 20-place table of the mantissas of all numbers from one to one million. Table I must be carried to 25 digits, and will contain about 83 entries; and this table must be permanently recorded and incorporated in the governor. The first number whose logarithm we want is 100,001. We enter this upon dial A, touch a button, and step back.

The governor then enters the number 100,001 on the lower dial of B, enters the number r_1 on the keyboard of B, and initiates the process of automatic division. If the first digit in the quotient is zero (as it will be in this case), B is stopped immediately and r_2 is substituted for r_1. This continues until an r is found which does not give zero for the first digit of the quotient (it will be r_{18}

in this case), and this time B is allowed to complete the division while the logarithm of this r is recorded in C. Next the quotient in the upper dial of B must be transferred to its lower dial. This continues till all the r's have been used. Then the number 100,001 and its logarithm found in machine C are printed. Next, dial A moves up to 100,002, and the process starts over.

It would take quite a while to complete the table. Suppose that each digit in a quotient can be obtained in one second, so that 25 seconds are required to get a quotient of 25 digits. Suppose, furthermore, that 40 divisions are needed and 43 values of r are rejected in calculating one logarithm. At this rate we should secure perhaps two logarithms each hour, and the whole table in 50 years, with no time out for repairs. Of course a hundred robots could divide up the work and finish the job in six months.

Antilogarithms can be found from Table I by reversing the process. It is left to the reader's imagination to conceive another Robot to calculate antilogarithms.

TABLE I

n	$r_n = 10^{1/2^n}$	$\log_{10} r_n = 1/2^n$
1	3.1622777−	.50000000
2	1.7782794+	.25000000
3	1.3335214+	.12500000
4	1.1547820−	.06250000
5	1.0746078+	.03125000
6	1.0366329+	.01562500
7	1.0181517+	.00781250
8	1.0090350+	.00390625
9	1.0045074−	.00195312+
10	1.0022511+	.00097656+
11	1.0011249+	.00048828+
12	1.0005623+	.00024414+
13	1.0002811+	.00012207+
14	1.0001405+	.00006104−
15	1.0000703−	.00003052−
16	1.0000351+	.00001526−
17	1.0000176−	.00000763−
18	1.0000088−	.00000381+
19	1.0000044−	.00000191−
20	1.0000022−	.00000095+
21	1.0000011−	.00000048−
22	1.0000005+	.00000024−
23	1.0000003−	.00000012−
24	1.0000001+	.00000006−
25	1.0000001−	.00000003−

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Mathematics of Business and Finance. By W. B. Dyess and R. O. Gilmore. New York, McGraw-Hill Book Company, Inc., 1942. 10+221+8+214 pages. $3.50.

The typography and arrangement of material on the pages of this book is excellent. The first 96 pages are devoted to a review of arithmetic, algebra, logarithms, and graphic representation. After the foundation has been made secure the authors take up the usual topics of a course in the mathematics of finance in the next 120 pages. With the text proper is bound the "Compound Interest and Annuity Tables" by F. C. Kent and M. E. Kent. These tables are printed on a paper that minimizes the eyestrain involved in their use.

In the chapter on arithmetic are found some rapid methods of multiplication and division and the methods of checking calculations by casting out nines and elevens. Throughout the book the explanations are clear and illustrative problems are solved in detail by three methods—binomial series, logarithmic computation, and the use of tables of compound interest and annuities. This emphasis on the three procedures which may be used in handling problems involving compound interest is the most admirable feature of the book. In using this text a teacher might well take up the chapter on graphic representation before the chapter on logarithms in order that the class might have the benefit of the discussion of linear interpolation from a graphical standpoint prior to its utilization in the study of logarithms.

The authors have unfortunately used eight or ten place logarithmic tables in many of the computations involved in the illustrative problems (Kent's tables are six place). The results in most, but not all of these problems, have been rounded off to six significant figures, but these answers cannot always be duplicated with the tables in the book. In the chapter on life insurance values for D_x and N_x with ten significant figures are said to be obtained from Kent's tables which give them with six significant figures. A more serious criticism may be made of the author's use of linear interpolation to obtain the value of 1.0325^{20} (page 120) and 1.03125^{-20} (page 125) to ten places of decimals. Three decimals are the best one can hope for in these cases. If logarithmic methods are to be avoided, use should be made of Newton's interpolation formula which is discussed on pages 8–11 in Kent's tables.

The reviewer cannot understand the author's lack of use of the theorem on logarithms of powers. Two examples are cited: In evaluating $S=2{,}478.25(1.0125)^{40}$

on page 120, the text reads, "To find S, we add the logarithms of 2,478.25 and $(1.0125)^{40}$. The value of $(1.0125)^{40}$ is found in the tables on page . . . ." On page 121 the value of $(1.03)^{125}$ is obtained by looking up the values of $(1.03)^{100}$ and $(1.03)^{25}$, finding their logarithms, adding, and then finding the antilogarithm.

In conclusion it may be said that this book will be found to be a very usable text since students pay little attention to the details of illustrative problems.

L. A. DYE

First Year College Mathematics. By C. C. Richtmeyer and J. W. Foust. New York, F. S. Crofts and Co., 1942. xi+461 pages $3.25.

As stated by the authors "This book presents in one volume the essentials of a first year's work in college mathematics. It includes a thorough coverage of algebra, trigonometry, and plane analytics, and an introduction to the fundamental concepts of differentiation and integration."

The notions of the calculus are introduced early, in Chapter III, by means of "average rate of change" and "exact rate of change". Here the reviewer was quite distressed by the authors' complete ignoring of that "fundamental concept of differentiation", the limit. The word does not even appear in the book. This would not be a serious omission if the notion were present. On p. 41, where differentiation is first discussed, appears the phrase "the interval Δx approaches zero" with nothing like an adequate explanation of its meaning. It is unthinkable, to the reviewer, that the Δ-process be presented to a student without mention of the idea of limit. As far as he can see the word "approaches", wherever it is used by the authors, can be and will be replaced by the naïve reader by "equals". Such writing will lead the student to the belief that evaluation of the limit, by whatever name the process may be called, and substitution are identical. Indeed, when we get to p. 126 and a discussion of the differentiation of irrational functions (marked *optional* in the text) the reader is suddenly instructed to rationalize the numerator of the difference-quotient; he must certainly wonder why. Previously the Δx in the numerator just "canceled" the one in the denominator. Here he can see no reason for not obtaining that hardy perennial, $D_x y = 0/0$.

Except for this gap the authors have written a simple, clear, and well-unified text. They have written their book so that the function concept is the unifying principle. A perusal of the chapter headings is indicative of their objectives. The headings are: I. A Review of Elementary Algebra; II. Functions and Graphs; III. Derivatives and Antiderivatives; IV. First Degree Functions and Linear Equations; V. Second Degree Functions and Quadratic Equations; VI. General Polynomial Functions and Polynomial Equations; VII. Fractional Functions and Equations; VIII. Irrational Functions and Equations; IX. Trigonometric Functions of Acute Angles; X. Common Logarithms; XI. Solution of Right Triangles by Logarithms; XII. Trigonometric Functions of General Angles; XIII, Solution of Oblique Triangles; XIV. Radian Measure, Inverse Functions,

Graphs; XV. Systems of Equations; XVI. Introduction to Analytic Geometry; XVII. The Straight Line; XVIII. The Circle; XIX. Parabola, Ellipse, and Hyperbola; XX. Transformation of Coordinates; XXI. Tangents and Normals; XXII. Polar Coordinates; XXIII. Parametric Equations; XXIV. Progressions; XXV. Permutations, Combinations, Probability; XXVI. Mathematical Induction and Binomial Theorem; XXVII. Complex Numbers; XXVIII. Natural Logarithms and Exponentials.

An excellent feature of the book is the series of "Self-tests" which appear at the end of each chapter. These consist of questions which lead to a complete review of the chapter. Answers to these tests are given at the back of the book.

Many of the chapters also have, as concluding sections, optional topics which the teacher with the above-average class will want to include in the course. Some of these are unique in a text of this sort, such as the section on Musical Sounds. There is an abundance of illustrative examples and excellent exercises. Answers are given to the odd-numbered ones.

In the opinion of the reviewer First Year College Mathematics is a good text and merits careful consideration by teachers. L. L. LOWENSTEIN

A Diagnostic Study of Students' Difficulties in Mathematics in First Year College Work. By E. N. Boyd. (Contributions to Education, No. 798.) New York, Bureau of Publications, Teachers College, Columbia University, 1940. 152 pages. $1.85.

A large number of college graduates have told me that the instruction they had in college mathematics was inferior to that which they received in preparatory school. For many years preparatory school teachers have given diagnostic tests to determine a student's difficulties and then proceeded to help the student over these hurdles. On the other hand, most college instructors have taken the attitude—here's the subject, understand it if you can and if you are so dumb you can't, get out! It is indeed a pleasure to find one college instructor who is interested in determining what difficulties her students have. This book deals with tests given freshmen in the general course in mathematics at Hunter College. The tests cover trigonometry, analytic geometry and the elements of differential calculus. It is well worth the while of every teacher of college freshmen to read these tests and study the results, for, by so doing, I am confident he would improve his teaching. F. M. MORGAN

The Reading of Verbal Material in Ninth Grade Algebra. By Margaret Grace McKim. Bureau of Publications, Teachers College. Columbia University, 1941. 8+133 pages. $1.60.

This is a study based on the realization of the importance of the reading problem to the child and the necessity of specific training in a special field like algebra.

There is a minute description of the construction and adaptation of tests

given, in the final experiment, to one hundred twenty New York City pupils of the ninth grade. The tests cover two areas—the reading of explanatory passages and the reading of problems. The tests are reproduced.

The results are then tabulated in juxtaposition with those obtained from the Terman Group Test of Mental Ability, the New Stanford Reading Test, the Orleans Algebra Prognosis Test, and the Mid-Term and Final examinations given in the school.

The significance of the intercorrelations are a problem of statistical measurement. Dr. McKim finds, common to all the tests and to the tests in groups, elements which are significant in determining variation of reading efficiency in passing from non-algebraic to algebraic material and from one kind of algebraic material to another, also in determining relationship between reading ability and achievement in algebra.

MILDRED W. HAFF

Early Military Books in the University of Michigan Libraries. By Thomas M. Spaulding and Louis C. Karpinski. The University of Michigan Press, 1941. 45 pages and 37 plates. $2.00.

This is a compilation of titles of books published before 1800 on military art and science (military history excluded), that can be found in the University of Michigan Library. There is an index of military books by mathematicians and mathematical works with sections on military science.

The plates reproduce the titles pages of some of the books listed.

MARK KAC

NEW BOOKS RECEIVED

Non-euclidean Geometry. By H. S. M. Coxeter, Mathematical Expositions, No. 2, Toronto, University of Toronto Press, 1942. 15+281 pages. $3.25.

Differential and Integral Calculus. By H. M. Bacon. New York and London, McGraw-Hill Book Co., 1942. 7+771 pages. $3.75.

Calculus. By G. E. F. Sherwood and A. E. Taylor. New York, Prentice-Hall, Inc., 1942, 14+503 pages. $3.75.

Analytic Geometry and Calculus. By H. B. Phillips. Cambridge, Mass., Addison-Wesley Press, 1942. 10+490 pages. $5.00.

Metric Methods in Finsler Spaces and in the Foundations of Geometry. By H. Busemann. (Annals of Mathematics Studies, No. 8.) Princeton, Princeton University Press; London, Humphrey Milford and Oxford University Press, 1942. 247 pages. $3.00.

Topics in Topology. By S. Lefschetz. (Annals of Mathematics Studies, No. 10.) Princeton, Princeton University Press; London, Humphrey Milford and Oxford University Press, 1942. 139 pages. $2.00.

Finite Dimensional Vector Spaces. By P. R. Halmos. (Annals of Mathematics Studies, No. 7.) Princeton, Princeton University Press; London, Humphrey Milford and Oxford University Press, 1942. 5+196 pages. $2.35.

Analytic Topology. By G. T. Whyburn. (American Mathematical Society Colloquium Publications, vol. 28.) New York, American Mathematical Society, 1942. 10+278+1 pages. $4.75.

An Outline of Probability and Its Uses. By M. C. Holmes. Minneapolis, Burgess Publishing Co., 1936. 8+119 pages. $1.59.

Mathematics in Human Affairs. By F. W. Kokomoor. New York, Prentice-Hall, Inc., 1942. 9+754 pages. $5.35.

A Mathematics Refresher. By A. Hooper. New York, Henry Holt and Co., 1942. 10+342 pages. $1.90.

The Methodology of Pierre Duhem. By A. Lowinger. New York, Columbia University Press, 1941. 184 pages. $2.25.

Mathematics of Modern Engineering. By E. G. Keller. Volume II. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd., 1942. 12+309 pages. $4.00.

Algebra. By P. L. Evans. New York, Ginn and Company, 1941. 8+126 pages. $1.25.

Plane Trigonometry with Tables. By P. L. Evans. New York, Ginn and Company, 1941. 8+84 pages. $1.25.

Calculus. By P. L. Evans. New York, Ginn and Company, 1941. 8+126 pages. $1.25.

Plane Trigonometry. By R. E. Heinman. New York and London, McGraw-Hill Co., Inc., 12+167 pages. $2.00.

Spherical Trigonometry and Tables. By W. A. Granville, P. A. Smith and J. S. Mikesh. New York, Ginn and Company, 1942. 18+60+4+43 pages. $1.25.

Plane Trigonometry, Solid Geometry and Spherical Trigonometry with Tables. By W. W. Hart and W. L. Hart. Boston, D. C. Heath and Co., 1942. 8+280 +124 pages. $2.60.

Essentials of Plane and Spherical Trigonometry. By A. H. Sprague. New York, Prentice-Hall Inc., 1942. 9+169 pages. $1.35.

Blueprint Reading at Work. By W. W. Rogers and P. L. Welton. New York, Silver, Burdett Co., 1942. 8+136 pages. $1.28.

Shop Mathematics at Work. By P. L. Welton and W. W. Rogers. New York, Silver Burdett Co., 1942. 4+204 pages. $1.56.

Mathematics for Electricians and Radiomen. By N. M. Cooke. New York and London, McGraw-Hill Book Company, Inc., 1942. 8+604 pages. $4.00.

Fundamentals of Radio. By E. C. Jordan, P. H. Nelson, W. C. Osterbrook, F. H. Pumphrey and L. C. Lynne. E. V. Everitt, Editor. New York, Prentice-Hall, Inc., 1942. 12+400 pages. $5.00.

Table for Sine and Cosine Integrals for Arguments from 10 *to* 100. New York, Works Projects Administration, 1942. A. N. Lowan, Technical Director. 32+189 pages. $2.00.

Original Tables to 137 *Decimal Places of Natural Logarithms for Factors of the Form* $1 \pm n \cdot 10^{-p}$, *Enhanced by Auxiliary Tables of Logarithms of Small Integers*. By H. S. Uhler. New Haven, Conn., Horace S. Uhler, 1942. 120 pages.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Allegheny College, Meadville, Pa.

REQUESTS FROM CLUBS

In our circular letter of May 20, 1942 we asked "What material, in addition to club reports, do you consider most suitable and valuable for inclusion in the limited publication space of this department of the MONTHLY? Do you know of any papers by students or faculty which might be considered for publication in this department?" Nine replies were received to the first question, but no suggestions were received in answer to the second. Topics with references, and bibliographies were requested by several. They would be welcomed by the editor. Others requested papers suitable for use on mathematics club programs. Another request was for a solicited list of some students who would like to carry on private correspondence on mathematical topics of mutual interest with students from other clubs. Names of such students, with the topics on which they would like to correspond, may be sent to the editor. A request for suggestions of mathematical games for informal parties was also received. We hope the suggestions which follow will prove useful for such occasions.

A MATHEMATICAL PAUL JONES

How to run a *Paul Jones* dance so that no one dances with the same partner twice is easily solved by a mathematician if the number of couples is $p-1$, where p is a prime. Assign numbers from 1 to $p-1$ to the men and also to the ladies. After one round with the partner of the same number, require the gentlemen to double their original numbers, and subtract p if the result exceeds p. Each should then have a new partner. On the third round the gentlemen should multiply their original numbers by three, and on the fourth by four, etc., subtracting multiples of p in each case. Each time the men should find new partners, until each has danced with every lady. If two men claim the same lady at the same time, she should check their arithmetic and pick the correct one.

A variant of the same idea is the following game designed to test speed in multiplication and to provide a little amusement at informal mathematical parties. Let 12 (or $p-1$) persons be seated around a circle on seats which are assigned numbers from 1 to 12 (or $p-1$). The Number One man (who is considered to be at the bottom of the ladder) picks a key number n, less than 13 (or p), and announces it. He then calls out a multiplier m less than 13 (or p). Everyone then stands *except* the one whose seat number s is such that $mn-s$ is divisible by 13 (or p). If that man also stands, he goes to the bottom of the ladder, all those below him move up one seat, and his old seat number s becomes the new key number n. Otherwise everyone sits, and the Number One man tries

another multiplier. After a specified time, the contestant who occupies the highest numbered seat is declared the winner. If the number of contestants is not of the form $p-1$, p prime, certain seats can be turned backwards at the start and left vacant, although they are assigned seat numbers. Then if Number One calls a multiplier m such that mn corresponds to a blank seat, he is declared out of the game, and all seats (including vacant seats) are renumbered, starting with the lowest occupied seat as Number One. The play proceeds until the time is up or until all but one of the contestants are eliminated.

CLUB REPORTS 1941–42

Pi Mu Epsilon, University of California at Los Angeles

Meetings were held once each month during the school year, at which the following papers were presented: *Functions of positive derivatives*, by Dr. A. E. Taylor, *Mathematics in oceanography*, by Mr. Robert Gordon, *Mathematical quotations from nonmathematical sources*, by Dr. Max Zorn, *Statistical methods of weighting examination questions*, by Dr. Paul Hoel, *Summation of m^{th} powers of the first N integers*, by Mr. William Gustin. Six new members were accepted during the year. The pledges for the fall and for the spring were presented at regular meetings. The fall initiation was held at the Helen Mathewson Club on Nov. 14, 1941; the spring initiation was held at the home of Dr. Paul Hoel on April 25, 1942; both in the evening. At each initiation the pledges were required to present a program lasting half an hour. Donald Wall was this year's winner of the ten dollar prize in the annual calculus contest sponsored by the organization. Officers for 1941–42 were as follows. Director, Robert White (Sept. to Feb.), Melvin Henry (Feb. to June); Vice-Director, Dorothy Stanley; Secretary, Jane Zartman; Treasurer, Wendell Mason; Program chairmen, Joel Ginsberg, Clay Perry, Gordon Overholtzer; Scholarship committee, Taffee Tanimoto; Sponsors, Dr. A. E. Taylor and Dr. Ralph Byrne.

Pi Mu Epsilon, University of Georgia

The *Georgia Alpha Chapter of Pi Mu Epsilon* sends greetings and reports a successful year. The chapter held regular meetings twice each month. Included among the topics discussed at the program meetings were *Induction into the army* by Captain H. A. Robinson of Ft. McPherson, *Mathematics in art—the Italian, the Chinese, and the Egyptian methods of projection* by Professor Jean Charlot of the Art Faculty, *Discussion and demonstration of minimum surfaces, Extracting cube roots with calculating machines, Graphical methods—new applications, Solution of practical problems presented to the group.* On April 16 at Memorial Hall the annual initiation banquet was held. Fifteen new members were initiated and the new officers were elected. Officers for 1941–42 and 1942–43 were as follows. Faculty Director, Dr. D. F. Barrow (41–42), Professor Forrest Cumming (42–43); President, J. A. Johnson Jr. (41–42), Gladys Feagin (42–43); Vice-President, Ann H. Davis (41–42), J. R. Mangham (42–43); Secretary-Treasurer, A. T. Harmon (41–42), Robert Collat (42–43); Corresponding secretary, Iris Callaway (41–43).

Pi Mu Epsilon, University of Arkansas

Pi Mu Epsilon held regular meetings during the year with discussions on various topics. Dr. Harrison Hale of the Chemistry Department gave the principal address on *The value of mathematics* at the annual fall banquet. Colonel Neilson gave his views on *The graveness of the war situation* at the spring banquet. There was a total of forty members during the year of whom thirteen were initiated the second semester. Several members bought keys and watch charms. Students who were initiated at the spring banquet were asked to be present at the annual Honors Day program, when the names of the new pledges were announced. The annual spring picnic was a success in spite of the threatening rain. New officers elected for 1942–43 were: Director, V. W. Adkisson; President, Francis Strabola; Secretary, Louise Williams; Treasurer, John Jacks.

Pi Mu Epsilon, University of Pennsylvania

Five talks were given as follows: *Breaking the law, upholding the law* by I. J. Schoenberg, *The Waring problem* by William Turanski, *Lissajou curves* by Fred Orttung, *Double star orbits* by P. M. Witman, *The observational approach to the problem of stellar multiplicity* (with slides) by Peter van de Kamp, director of the Sproul Observatory, Swarthmore College. Prizes were given in a problem contest to William Turanski and Tong Hing. Fred Orttung received a prize as the best student speaker. The initiation banquet was held in the Benjamin Franklin room of Houston Hall in January with Professor C. J. Rees of the University of Delaware as guest speaker. Officers for 1941–42 were President, Morton Brown; Secretaries, Ruth Tobias and Eli Perry.

Mathematics Club, Wellesley College

Meetings this year were informal; no real papers were presented. Rather, students gave short talks on the following topics: *Women in mathematics*; biographical talks on *Sir Isaac Newton, Leibniz, Archimedes*; the applications of mathematics in teaching, architecture, economics, and geology; and impossibilities such as five-sided squares. Source material for these talks consisted of standard reference books such as Bell's *Men of Mathematics*. The club also produced a play by Janet Brown '35, written as a mathematical burlesque of *Romeo and Juliet* and entitled *Ratio and Jacobean, a Trajectory*. It was not very serious, and we had a great deal of fun with it. The club report was written by the retiring secretary, Ruthven Tremain. Officers for 1942–43 are: President June Nesbitt '43, Vice-President Betty Ann Wilson '43, Treasurer Martha Adams '43, Secretary Phyllis Fox '44, Junior Executive Elizabeth Bird '44, Faculty Adviser, Miss Lennie Copeland (Chairman).

Kappa Mu Epsilon, Drake University

The *Mathematics Club of Drake University* became the Iowa Beta Chapter of *Kappa Mu Epsilon* on May 27, 1940. Officers for 1941–42 and 1942–43 were elected as follows. President, Bob Goss (41–42), Bob Lambert (42–43); Vice-President, Norman Landess (41–42), Bob Hansen (42–43); Secretary, Julia Rahm (41–43); Treasurer, Bob Lambert (41–42), Alex Smotkin (42–43); Pledge Master, Earl Carlson (41–42), John McKiernan (42–43); Publicity, Don Johnson (42–43). Professor Floy Woodyard acted as Faculty Adviser. The program for the year included the following topics. *Mathematics for the Navy* by Lieutenant Commander Lauder, *Mathematics in industry and the Telephone Company* by Mr. McLellan of the Telephone Company, *Mathematics in the present emergency and in aviation* by Professor Mehlin.

Mathematics Club, Milwaukee-Downer College

The two papers presented were *Dynamic symmetry* by Professor E. R. Beckwith, and *Taxes and government financing* by Miss Ferol Bosl. The remaining meetings of the year were devoted to instruction in the use of the slide rule, in view of the need for this skill in defense work. Instruction was open to anyone in the college who cared to receive it. Contributions of work and money were made to the Red Cross campaign. Officers for 1942–43 are as follows. President, Dorothy Puelicher; Secretary-Treasurer, Dorothy Rodgers.

Kappa Mu Epsilon, Kansas State Teachers College at Emporia

A total undergraduate membership of forty-five students participated in eight meetings throughout the year, which alternated with meetings of the Mathematics Club. Discussions of the scientific work of the General Motors Corporation, of the importance of mathematics in national defense, of the use of spherical trigonometry in aviation and navigation, and of meteorology were on the program. Officers for the summer were: President Pascal, Eugene Etter; Vice-President Gauss, Kay Wilch; Secretary-Treasurer, Mrs. Ruby Norris. Officers elected for 1941–42 were: President Pascal, Warren Burns; Vice-President Gauss, Alfred Freeman; Secretary Eratosthenes, Frances Breneman; Treasurer Bhaskara, Rosemary Haslouer; Historian Ahmes, Mary Townsend; Corresponding Secretary Descartes, Professor C. B. Tucker; Sponsor Thales, Dr. O. J. Peterson.

Mathematics Club, Kansas State Teachers College at Emporia

The place of mathematics and physics in defense by Dr. Cram, *How to study for finals* by a round table consisting of Lester Meisenheimer, Wilbur Schoof, Carter Sigel, and Frances Breneman, *The light weight champion* (magnesium) *in the present emergency*, by Dr. Blackman, *Prices and free enterprise* by Dr. Pickett, and *Games of chance*—these were the features of a program of eight meetings which wound up with a picnic on May 6. Officers were: President, Mary Alice Anderson; Vice-President, Carter Sigel; Secretary-Treasurer, Frances Peterson.

Kappa Mu Epsilon, Upsala College

The Alpha Chapter of New Jersey closed a very successful year with the annual banquet, initiation of five new members, and the installation of next year's officers. Nine meetings were held during the year, for seven of which the students worked out their own projects and researches. Topics included were: *Fundamentals of drawing from a machinist's standpoint*, by Joseph May, *Sidelights on the development of trigonometry with a note on Mollweide's theorm*, by Lillian Meisel, *The derivation of the normal probability curve*, by Anne Zmurkiewicz, *Mathematical paradoxes with a note on "Geometry on wheels,"* by Marjorie Nicoll, *How mathematics makes money for the worker*, by Bernard Morrow, *Sir Isaac Newton*, by Edward Cohen. Two outside speakers addressed the chapter. Professor Virgil Mallory of Montclair State Teachers College gave an elucidating talk on the *Mathematics of defense* at mid year, and Mr. Edward Molina, research statistician with the Bell Telephone Company, gave a talk on *Probability* at the annual banquet on May 22. Some insight into the history of mathematics was provided at every other meeting by having each officer discuss in turn the life and contribution of his chapter name-sake. Thus Secretary Abel, Vice-president Appollonius, Treasurer Fibonacci, and Secretary Descartes spoke to us from the distant past to the living present. Officers installed on May 22 are: President, Phyllis Gustafson; Vice-President, Edward Cohen; Secretary, Marjorie Nicoll; Treasurer, Lillian Meisel; Historian, the retiring president Anne Zmurkiewicz.

Mathematics Club, Brown University

The club held six monthly meetings through the year, which were scheduled in advance and announced in a printed program before the first meeting. Two students spoke at each of four meetings, after preparing their talks in consultation with a member of the faculty. Their topics were *Short cuts in arithmetic* by Doris Keighley, *The duodecimal system* by R. L. Johnson, *Alignment charts* by R. P. Gosselin, *Areas by machine*, by Dieter Kurath, *Magic squares* by Arline Major, *Chess, caves, and cannibals* by E. J. Bernier, *Linkages* by W. F. Jones, Jr., *Mechanical devices for computation* by Granino Korn. At the January meeting Professor George Polya spoke on *Counting chemical compounds*, and at the March meeting Lieutenant Commander E. T. Goyette spoke on *The mathematical inaccuracy of navigation*. A group picture was taken after this meeting. The club picnic scheduled for May 7 was canceled because of the accelerated program and gasoline rationing Officers for the year were as follows. Committee on program: Professor J. S. Frame, *faculty representative*; Tamara Bachman, R. L. Johnson, Dieter Kurath, Ellen Swanson, Paul Tamarkin. Committee on arrangements: Granino Korn, *chairman*; E. J. Bernier, J. E. Cook, Jr., Ruth Cunningham, W. F. Jones, Jr., Doris Keighley, Arline Major.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 546. *Proposed by W. E. Buker, Pittsburgh Public Schools*

Show that $\frac{1}{5}x^5+\frac{1}{3}x^3+\frac{7}{15}x$ is an integer for every integral value of x.

E 547. *Proposed by V. Thébault, San Sebastián, Spain*

A diameter d of the circumcircle of an equilateral triangle ABC cuts the sides BC, CA, AB in points D, E, F. Prove that the Euler lines of the three triangles AEF, BFD, CDE form a triangle symmetrically equal to ABC, the center of symmetry lying on d.

E 548. *Proposed by R. V. Heath, Wall St., New York City*

Find a perfect square of seven digits with all digits even and positive. Show that the digits of a perfect square (>9) cannot be all *odd*.

E 549. *Proposed by L. M. Kelly, U. S. Coast Guard Academy*

The face planes of a proper tetrahedron intersect a circumscribed quadric cone in four parabolas. What conditions are thus imposed on the cone?

E 550. *Proposed by D. H. Browne, Buffalo, N. Y.*

Prove that, for every positive even value of m, the sum of the mth powers of alternate numbers from 1 or 2 up to n is a polynomial function of n (of the same form whether n be odd or even).

SOLUTIONS

An Automorphic Number

E 506 [1942, 120]. *Proposed by R. V. Heath, Wall St., New York.*

Show that, for every positive integer n, the last $n+9$ digits of

$$90625^{2^n}$$

form an automorphic number. [See 1941, 407].

Solution by E. P. Starke, Rutgers University

Since 90625 is divisible by 5^5, the number

$$P = 90625^{2^n}$$

is divisible by $5^{5\cdot 2^n}$ and hence by 5^{9+n} (as $5\cdot 2^n \geqq 9+n$). Furthermore, $P-1$ is divisible by 2^{9+n}, as is easily established by induction from the case when $n=0$, using the relation

$$(k2^{9+n}+1)^2 = (k^2 2^{8+n}+k)2^{10+n}+1.$$

It follows that P^2-P is divisible by both 5^{9+n} and 2^{9+n}, hence by 10^{9+n}. Finally a may be chosen so that the number

$$M = P - a\cdot 10^{9+n}$$

consists of the last $9+n$ digits of P. The difference

$$M^2 - M = P^2 - P - 2a.10^{9+n}P + a^2\cdot 10^{18+2n} + a\cdot 10^{9+n}$$

is seen to be divisible by 10^{9+n}, and thus the last $n+9$ digits of M^2 are the digits of M, as required.

Editorial Note. This problem suggests the following practical rule for successively computing the digits of an automorphic number (from right to left): If N is the odd automorphic number of m digits, then the last $m+1$ digits of N^2 form an automorphic number with one extra digit.

Proof. We have $N^4-N^2=\frac{1}{2}(N+1)\cdot 2N\cdot(N^2-N)$.
But N is an odd multiple of 5, and $N^2-N\equiv 0 \pmod{10^m}$. Hence

$$N^4 - N^2 \equiv 0 \pmod{10^{m+1}}.$$

We can now find both the automorphic numbers. For, if N is the odd one, the even one is 10^m-N+1.

An Extension of E 467 to Three Dimensions

E 507 [1942, 120]. *Proposed by V. Thébault, San Sebastián, Spain*

In an orthocentric tetrahedron with orthocenter H and circumcenter O, show that the radical planes of the circumsphere with the respective spheres whose diameters are the four medians, meet the Euler lines of the corresponding faces in four points lying in a plane perpendicular to OH. (The *medians* of a tetrahedron join its vertices to the centroids of the opposite faces.)

Solution by L. M. Kelly, U. S. Coast Guard Academy

Label the four points P_1, P_2, P_3, P_4. It is evident that if we prove that planes through these points perpendicular to OH all meet OH in the same point Q, the proposition will be established. We first recall the following known properties of an orthocentric tetrahedron $A_1A_2A_3A_4$:

(1) the four altitudes are concurrent,

(2) the centroid G is the midpoint of OH,

(3) the product of the segments into which the orthocenter divides the altitudes is the same for all four altitudes,

(4) if G_1 is the centroid of the face $A_2A_3A_4$, then $3GG_1=GA_1$.

Now let H_1 and O_1 be the orthocenter and circumcenter of $A_2A_3A_4$, and let M_1 be the midpoint of A_1G_1. In the plane OHA_1, P_1 may be determined by drop-

ping a perpendicular from A_1 on OM_1 and producing it to meet O_1H_1. To determine Q it is only necessary to drop a perpendicular from P_1 on GH. Draw G_1H and note that it is parallel to OM_1, and hence perpendicular to AP_1. Thus H is the orthocenter of the triangle $A_1G_1P_1$. Finally, if P_1H is drawn meeting A_1G_1 in R_1, we see that the triangles GHR_1 and P_1HQ are similar, whence

$$GH \cdot HQ = R_1H \cdot HP_1 = H_1H \cdot HA_1,$$

which is constant (*i.e.*, the same as when P_1 is replaced by P_2, P_3, or P_4).

Also solved by the proposer.

Perfect Bridge Hands

E 508 [1942, 120]. *Proposed by R. K. Allen, Montpelier, Vermont*

How many bridge hands are there where all thirteen tricks can be taken at no trump regardless of the distribution of the cards? It is assumed that declarer will always play his highest cards and not intentionally lose any tricks. (Cf. E 448.)

Solution by H. W. Norton, University of Chicago

In order that a bridge hand may take all tricks at no trump regardless of the distribution of the remaining cards, it is necessary that that hand hold all the aces; the nine other cards may be variously distributed. For example, if the nine are all in one suit, there are three cards in that suit outstanding, and it is necessary that the nine should include the King and Queen and any seven of the remaining cards. Thus there are $4\binom{10}{7} = 480$ such hands, since the nine may be in any one of the four suits. The complete list of posibilities with their respective frequencies is as follows:

9—0—0—0	480	5—2—2—0	12
8—1—0—0	1512	5—2—1—1	12
7—2—0—0	672	4—4—1—0	12
7—1—1—0	672	4—3—2—0	24
6—3—0—0	84	4—3—1—1	12
6—2—1—0	168	4—2—2—1	12
6—1—1—1	28	3—3—3—0	4
5—4—0—0	12	3—3—2—1	12
5—3—1—0	24	3—2—2—2	4

The total is 3756.

It may be noted that it is necessary to hold consecutive cards below the Ace in any suit in which fewer than seven cards (including the Ace) are held. In criticizing the frequency with which so-called "perfect" bridge hands are reported in the press (*London Times*, May 15, 1939) I called attention to this new type of hand, which is perfect in that it shows before the bidding that every trick can be taken. Though it is about three years since I called attention to this type of hand, and though the ordinary "perfect" hand is reported several times each year, I have not yet seen mention of this hand which is perfect from the

trick-taker's point of view. Yet it is nearly a thousand times as frequent, with proper shuffling; and (according to Culbertson) imperfections of shuffling should make it relatively more frequent.

Also solved by W. E. Buker and the proposer.

The Square Bedroom and the Music Room

E 509 [1942, 121]. *Proposed by the late J. E. Trevor, Cornell University*

An architect is designing a house for his client's seventy-five foot lot. The "square bedroom" is to have a square floor, and it will contain an ordinary double bed and other furniture. The owner specifies that the music room shall be two feet longer than it is wide, and that its floor-area in square feet shall be three times that of the square bedroom. It is also specified that the widths of the two rooms shall be integer numbers of feet. Find these widths.

Solution by W. E. Buker, Pittsburgh Public Schools

Let $s \times s$ and $(t-1)\times(t+1)$ be the dimensions of the bedroom and music room. Then we have to find integers s, t, satisfying $3s^2=(t-1)(t+1)$, or

$$t^2 - 3s^2 = 1. \tag{1}$$

This is an instance of the famous misnamed "Pellian Equation," which can be solved by several methods, *e.g.*, by continued fractions. We know that all solutions can be derived from the obvious solution

$$t_0 = 1, \qquad s_0 = 0$$

by means of the recurrence relations

$$t_{n+1} = 2t_n + 3s_n,$$
$$s_{n+1} = t_n + 2s_n.$$

Thus the simplest positive solutions are:

$$t = 2, 7, 26, 97,$$
$$s = 1, 4, 15, 56.$$

Clearly, the third solution alone satisfies the conditions of the problem: so the desired widths are 15 and 25.

Also solved by R. K. Allen, Paul Brock, William Douglas, Howard Eves, L. M. Kelly, C. C. Oursler, C. A. Rupp, E. P. Starke, and the proposer.

Stirling Numbers of the Second Kind

E 510 [1942, 121]. *Proposed by S. H. Gould, University of Toronto*

Given $a_{p,q}=a_{p,q-1}+qa_{p-1,q}$, $a_{p,1}=a_{1,q}=1$, $(p, q=1, 2, 3, \cdots)$, prove

$$(p-1)a_{p,q} = \sum_{k=1}^{p-1} a_{p-k,q}\left\{(q+1)^{k+1} - \binom{p+q}{k+1}\right\}.$$

Solution by the Proposer

Comparing the given recurrence relation with Bjorling's formula

$$C_n^{k+1} = nC_n^k + C_{n-1}^k$$

for the numbers $C_n^k = \Delta^n 0^k/n!$ (*Journal für die reine und angewandte Mathematik*, vol. 28, 1844, p. 284), we see that

$$a_{p,q} = C_q^{p+q-1}.$$

Thus the well-known formula

$$C_{m+1}^{n+1} = \sum_{k=m}^{n} \binom{n}{k} C_m^k$$

(Boole, *Finite Differences*, 1872 or 1926, p. 29) is equivalent to

$$a_{p+1,q+1} = \sum_{k=-1}^{p-1} \binom{p+q}{k+1} a_{p-k,q} = a_{p+1,q} + \sum_{k=0}^{p-1} \binom{p+q}{k+1} a_{p-k,q}.$$

On the other hand, by repeated application of the formula defining $a_{p,q}$ we obtain

$$a_{p+1,q+1} = a_{p+1,q} + \sum_{k=0}^{p-1} (q+1)^{k+1} a_{p-k,q}.$$

Hence, by subtraction,

$$\sum_{k=0}^{p-1} a_{p-k,q} \left\{ (q+1)^{k+1} - \binom{p+q}{k+1} \right\} = 0,$$

as desired.

The Whistling Bomb

E 511 [1942, 195 and 475]. *Proposed by W. E. Bleick, U. S. Naval Academy, Annapolis*

A whistle, which emits a sound of constant pitch, is attached to a bomb. An observer at a fixed point on the ground sees the bomb dropped from rest at an initial angular elevation B. Assume that the bomb falls with a constant acceleration g and hits the ground at a distance L from the observer. The observer hears a variable whistle pitch because of the Doppler effect. The train of sound waves of maximum apparent pitch leaves the bomb at the elevation at which the component of the bomb's velocity in the direction of the observer is a maximum. Find this angular elevation, and show that it approaches two-thirds of the initial elevation if the initial elevation is small.

Solution by Howard Eves, Syracuse University

Let A be the required angle of elevation, v the velocity of the bomb at that elevation, V the component of v in the direction of the observer, s the distance the bomb has fallen, and H the initial altitude. Then we have

$$v = gt, \qquad V = v \sin A,$$
$$s = \tfrac{1}{2}gt^2, \qquad \tan A = (H - s)/L.$$

Eliminating v, t, and s, we get

$$V^2 = 2g(H - L \tan A) \sin^2 A.$$

The condition for V^2 to be a maximum is

$$2(H - L \tan A) \sin A \cos A - L \sec^2 A \sin^2 A = 0.$$

Dividing by the non-zero factor $L \sin A \cos A$, we get

$$2 \tan B - 2 \tan A - \sec^2 A \tan A = 0,$$

or

$$\tan^3 A + 3 \tan A - 2 \tan B = 0. \tag{1}$$

By Cardan's formula, this cubic (with discriminant $\sec^2 B > 1$) has the single real root

$$\tan A = (\sec B + \tan B)^{1/3} - (\sec B - \tan B)^{1/3},$$

which gives the sought angle of elevation A.

If B (and therefore A) is small, (1) gives approximately

$$3 \tan A - 2 \tan B = 0,$$

or (again approximately)

$$A = \tfrac{2}{3}B.$$

Also solved by Adolph Barjansky, F. A. Butter, Jr., L. M. Kelly, E. P. Starke, and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers, would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4061. *Proposed by N. A. Court, University of Oklahoma*

A variable triangle has its vertices on three skew straight lines, and two of its sides meet a given plane in points lying respectively on two straight lines. Show that the points of intersection of that plane with the third side of the variable triangle are collinear.

4062. *Proposed by N. S. Mendelsohn, University of Toronto*

Show how the operation of multiplying by a real number may be expressed in terms of the operations of adding a real number and of reciprocating.

4063. *Proposed by H. S. M. Coxeter, University of Toronto*

In projective geometry the porism of triangles inscribed in one conic and self-polar for another is commonly proved by showing that if one such triangle exists, we can find another with one vertex at *any* given point on the first conic. This statement is easily seen to be valid in complex geometry. Discuss its possible failure in real geometry.

4064. *Proposed by V. Thébault, San Sebastián, Spain*

Given a tetrahedron $ABCD$: (1) Find the locus of points M such that the sum of the powers of the vertex A with respect to the spheres with diameters MB, MC, MD is constant. (2) Find the point M such that for the spheres with the diameters MA, MB, MC, MD the sum of the powers of a vertex with respect to the three spheres not passing through that vertex is the same for the four vertices. Show that the point M in this case is the symmetric of the centroid with respect to the circumcenter of the tetrahedron.

SOLUTIONS

Tetrahedron of Polar Planes

4004 [1941, 483]. *Proposed by N. A. Court, University of Oklahoma*

Given four spheres, let (E) be the tetrahedron formed by their centers, and (F) the tetrahedron formed by the four polar planes, for these spheres, of their radical center U. Prove that (i) if (E) admits U for its orthocenter, the same holds for (F); (ii) conversely, if (F) admits U for orthocenter, the same holds for (E).

I. *Solution by Howard Eves, Allen Academy, Bryan, Texas*

Let E_1, E_2, E_3, E_4 be the vertices of (E) and let F_1, F_2, F_3, F_4 be the corresponding vertices of (F).

(i) By hypothesis E_1U is perpendicular to $E_2E_3E_4$. Therefore E_1U is the radical axis of the spheres at E_2, E_3, E_4. But F_1, the common point of the polars of U for these three spheres, lies on the radical axis of the three spheres. Therefore E_1U passes through F_1. But we also have E_1U perpendicular to $F_2F_3F_4$, the polar of U for the sphere at E_1. Hence F_1U is an altitude of (F). Similar remarks hold for F_2U, F_3U, F_4U. Hence the first part of the theorem.

(ii) By hypothesis F_1U is perpendicular to $F_2F_3F_4$. Therefore F_1U passes through E_1. But F_1U is the radical axis of the spheres at E_2, E_3, E_4. Therefore F_1U is perpendicular to $E_2E_3E_4$. Hence E_1U is an altitude of (E). Similar remarks hold for E_2U, E_3U, E_4U. Hence the second part of the theorem.

Note: The corresponding theorem for the plane is similarly proved.

II. *Solution by the Proposer*

The point U is the center of a sphere (U), real, or imaginary, orthogonal to the four given spheres (A), (B), (C), (D). Since the two spheres (U), (A) are orthogonal, the polar plane of the center U of (U) for (A) is also the polar plane of the center A of (A) for (U). Similarly for the (B), (C), (D). Thus the two tetrahedrons $(E)=ABCD$, $(F)=PQRS$ are polar reciprocal with respect to the sphere (U).

Consequently the two propositions to be proved may be stated as follows. The polar reciprocal of an orthocentric tetrahedron for a sphere having the orthocenter of the tetrahedron for its center, is also an orthocentric tetrahedron having the same orthocenter as the given tetrahedron.

This proposition is fairly obvious. Indeed, using the same notations, the polar plane QRS of A for (U) is perpendicular to AU and therefore parallel to the plane BCD, and similarly for the other pairs of planes of the two tetrahedrons. Again, the pole P of the plane BCD for (U) lies on the perpendicular AU to BCD, and similarly for other pairs of corresponding vertices of the two tetrahedrons. Thus U is the homothetic center of the two tetrahedrons, hence the proposition.

Remark. Let p be the square of the radius of the sphere (U), and k, k' the squares of the radii of the polar spheres of the orthocentric tetrahedrons (E), (F). (See the proposer's *Modern Pure Solid Geometry*, art. 457.) If A', P' are the traces of the line APU in the planes BCD, QRS, we have

$$UA\cdot UA' = k, \qquad UP\cdot UP' = k', \qquad UA\cdot UP' = UA'\cdot UP = p,$$

hence $UA\cdot UA'\cdot UP\cdot UP'=kk'$, $UA\cdot UP'\cdot UA'\cdot UP=p^2$, and therefore $kk'=p^2$.

Moreover, we have

$$\frac{UA\cdot UA'}{UP\cdot UP'}=\frac{k}{k'}, \qquad \frac{UA\cdot UP'}{UA'\cdot UP}=\frac{p}{p}=1.$$

Now from the last equality we have $UA:UP=UA':UP'$, hence the homothetic ratio $UA:UP$ of the two tetrahedrons (E), (F) is equal to $\sqrt{k/k'}$.

Note. The analogue, for three circles, of the direct proposition is due to R. Tucker. *Educational Times*, Reprints, vol. 52 (1895), p. 28, Q. 12276.

Editorial Note. There is an actual tetrahedron (F) if U does not lie in a face of (E) and if the four given spheres do not have a point in common.

Perspective Tetrahedrons

4005 [1941, 483]. *Proposed by V. Thébault, San Sebastian, Spain*

An arbitrary plane (P) cuts the planes of the faces of the given tetrahedron $ABCD$ in four straight lines. The four planes through these straight lines perpendicular each to the corresponding face determine a tetrahedron $A'B'C'D'$. Show that: (1) The straight lines AA', BB', CC', DD' are concurrent. (2) The point of concurrency I is the center of one of the spheres tangent to the four

planes symmetric to (P) with respect to the faces of $ABCD$. (3) The point I remains fixed when (P) moves parallel to itself.

Editorial Note. Let $\alpha, \beta, \gamma, \delta$ denote the planes of the faces of the tetrahedron T, $\alpha', \beta', \gamma', \delta'$ the planes of the faces of T' where corresponding faces of T and T' intersect on a plane σ in a complete quadrilateral. Then the four straight lines each joining the corresponding vertices of T and T' meet in a point V, the center of perspectivity of T and T', and σ is called the plane of perspectivity, see Court's *Modern Pure Solid Geometry*, p. 21, and the solution of 3988 [1942, 409]. Let σ_1 be a plane parallel to σ, T_1 and T' have their corresponding faces parallel, and the corresponding faces of T_1 and T intersect on σ_1; then the vertex $(\alpha_1\beta_1\gamma_1)$ of T_1 lies on the straight line of V, $(\alpha\beta\gamma)$, $(\alpha'\beta'\gamma')$, and similarly for the other corresponding vertices. For, the figures $\alpha\beta\gamma\alpha'\beta'\gamma'\sigma$ and $\alpha\beta\gamma\alpha_1\beta_1\gamma_1\sigma_1$ have $(\alpha\beta\gamma)$ for homothetic center. This proves (1) and (3).

In this problem the corresponding faces of T and T' are perpendicular, and the planes of the faces of T'' are the symmetrics of σ with respect to the corresponding faces of T. The corresponding faces of the three trihedral angles $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$, $\alpha''\beta''\gamma''$ intersect on σ, and σ, α, α'', α' is a harmonic axial pencil of planes, and similarly for the two other sets of faces. It follows from this harmonic property that the three vertices $(\alpha\beta\gamma)$, $(\alpha'\beta'\gamma')$, $(\alpha''\beta''\gamma'')$ are on a straight line through the center of perspectivity I for T and T'. Also the distance of $(\alpha\beta\gamma)$ from σ is its distances from the faces of $\alpha''\beta''\gamma''$, so that the straight line $(\alpha''\beta''\gamma'')I$ is an axis of the trihedral angle $\alpha''\beta''\gamma''$, and similarly for the other trihedral angles of T''. Hence I is equally distant from the faces of T'', and the proof of (2) is complete. Part of the above argument is the same as that given by N. A. Court in his solution of 3988.

A Family of Curves

4006 [1941, 561]. *Proposed by P. D. Thomas, Norman, Okla.*

Find the equation of the family of curves, each curve having the property $LR = T^2$, where L is the distance of the tangent from the origin, R is the radius of curvature, and T is the length of the tangent from the point of contact to the y-axis.

I. *Solution by J. O. Hassler, University of Oklahoma*

If we introduce for L, R, and T their values

$$\frac{y - xy'}{\sqrt{1 + y'^2}}, \qquad \frac{(1 + y'^2)^{3/2}}{y''}, \quad \text{and} \quad x\sqrt{1 + y'^2},$$

respectively, the equation $LR = T^2$ becomes, after some reduction

$$x^2y'' + xy' - y = 0. \tag{1}$$

If we multiply (1) by $1/x^2$ and integrate, we get

$$y' + \frac{y}{x} = c,$$

which can be made an exact differential equation by clearing of fractions. The solution is

$$xy = c_1x^2 + c_2.$$

II. *Solution by R. K. Allen, Montpelier, Vt.*

Replace L, R, and T by their well-known and easily obtained differential expressions and the differential equation of the desired family is $x^2y''-xy'+y=0$. The general solution is $y=x(a \log x+b)$ where a and b are arbitrary constants. These curves are defined only for positive values of x, except when $a=0$, in which case the family is a pencil of straight lines through the origin, but then the expression LR degenerates into the meaningless $0 \cdot \infty$, so this particular case should be excluded. For all finite values of a and b $\lim_{x\to 0} y=0$, so we state that (0, 0) lies on every curve and is indeed the endpoint of the curve. Introduction of this single point brings in no difficulty for the equality $LR=T^2$ holds for all positive values of x and will still hold in the limit.

Solved also by N. R. Amundson, E. Fleisher, T. E. Mergendahl, R. W. Wagner, A. K. Waltz, and the proposer.

Editorial Note. Half of the solvers obtained the result in I and the rest that in II by using the negative of the expression for L in I. Some of the solvers introduced a change of independent variable. The process of integration in I may be used in II by writing $d[y'-y/x]/dx=0$ and then $d[y/x]/dx=a/x$.

A later solution by S. H. Lachenbruch considers both cases.

Four Associated Quadrilaterals

4007 [1941, 561]. *Proposed by J. W. Clawson, Ursinus College*

The straight lines l_i, $(i=1, 2, 3, 4)$, determine the complete quadrilateral Q. The four triangles determined by l_j, l_k, l_l have O_i, G_i for their circumcenters and centroids. The point P_i divides O_iG_i in the ratio $r:1$. Lines through these points parallel to l_i form the quadrilaterals Q_0, Q_g, Q_p. Prove that: (1) the four quadrilaterals are congruent; (2) the homothetic center of Q and Q_0 is the orthic center of $O_1O_2O_3O_4$; (3) the homothetic center of Q and Q_g is the mean center of $G_1G_2G_3G_4$; (4) the homothetic center of Q and Q_p divides the line joining the preceding points in the ratio $r:1$; (5) these homothetic centers all lie on the common mid-diagonal line of the four quadrilaterals.

Solution by the Proposer

Since it is well known that the four points O_i are concyclic, we shall take this circle for the unit circle in the complex plane, and take O_i to be the turn t_i. The focal point, F, which lies on this circumcentric circle, will be taken as the unit point. Then circles with centers O_i, O_j and radii O_iF, O_jF intersect at A_{kl}, one of the vertices of quadrilateral Q. The point A_{kl} is $t_i+t_j-t_it_j$, and the line l_i, determined by A_{ij} and A_{ik} is

$$z - t_jt_kt_i\bar{z} = t_j + t_k + t_l - t_jt_k - t_jt_l - t_kt_l,$$

where z is any point on the line and $\bar{z}$ is the complex number conjugate to z.

G_i is $t_i + (t_j + t_k + t_l)(1 - t_i)/3$ and P_i is

$$t_i + \frac{(t_j + t_k + t_l)(1 - t_i) \cdot r}{3 \cdot (r + 1)}.$$

The lines through O_i, G_i, P_i parallel to l_i are respectively

$$t_i z - s_4 \bar{z} = t_i^2 - t_j t_k t_l, \qquad t_i z - s_4 \bar{z} = t_i^2 - t_j t_k t_l + s_2(1 - t_i)/3,$$

and

$$t_i z - s_4 \bar{z} = t_i^2 - t_j t_k t_l + \frac{s_2(1 - t_i) \cdot r}{3(r + 1)}.$$

In these equations, s_m is used for the sum of the products of t_1, t_2, t_3, t_4 m at a time.

Hence, O_{ij}, G_{ij}, and P_{ij}, vertices respectively of Q_0, Q_ϱ, and Q_p are

$$t_i + t_j + t_k t_l, \qquad t_i + t_j + t_k t_l - s_2/3, \qquad t_i + t_j + t_k t_l - \frac{s_2 \cdot r}{3(r + 1)}.$$

Then the mid-points of lines joining A_{ij} to O_{ij}, G_{ij} and P_{ij} are $s_1/2$, $s_1/2 - s_2/6$ and $s_1/2 - s_2 \cdot r/6(r+1)$.

(1) From the symmetry of the last result, it follows that Q and Q_0, Q and Q_ϱ, and Q and Q_j are homothetic with ratio 1:1, *i.e.*, that they are all congruent, the homothetic centers being at the points indicated. Likewise (2), (3) and (4) are easily verified. Finally, (5) follows from the fact that each of these points, as also the middle points of $A_{ij}A_{kl}$, $O_{ij}O_{kl}$, $G_{ij}G_{kl}$, $P_{ij}P_{kl}$, *viz.*, $s_1/2 - (t_i t_j + t_k t_l)/2$, $s_1/2 + (t_i t_j + t_k t_l)/2$, $s_1/2 + (t_i t_j + t_k t_l)/2 - s_2/6$, and $s_1/2(t_i t_j + t_k t_l)/2 - s_2 \cdot r/6(r+1)$ all lie on the line $z - s_4 \bar{z} = s_1/2 - s_3/2$.

Editorial Note. The proposer gave the following definition of the orthic center of a cyclic quadrilateral:

The four straight lines each through the midpoint of a side and perpendicular to the opposite side of a cyclic quadrilateral are concurrent; and the point of concurrency, the orthic center, bisects the segment from any vertex to the orthocenter of the triangle with the other three points as vertices, see Clawson, *Annals of Mathematics*, vol. 20, no. 4, June 1919, p. 252.

The proof is easy; let $\mathbf{a}_i$ be the vector from the center of the circle to the vertex A_i, and $\mathbf{g}$ the vector to the centroid of the four vertices. The orthocenter of triangle $A_j A_k A_l$ is given by $\mathbf{a}_j + \mathbf{a}_k + \mathbf{a}_l$, and the midpoint of the segment from this orthocenter to A_i has the vector $(\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4)/2 = 2\mathbf{g}$. The vector from the midpoint of a side $A_i A_j$ to this latter point is $(\mathbf{a}_k + \mathbf{a}_l)/2$ which is perpendicular to the vector $A_l A_k$, or $\mathbf{a}_k - \mathbf{a}_l$.

A solution of this problem may be obtained by use of some of the results in 3839 [1939, 604], see also 3991 [1942, 550]. We shall give the results for (2). The homothetic center of (Q) and (Q_0) has the coordinates $a(1 + \sigma_2 + \sigma_4)/4$, $a\sigma_1/2$; the orthocenter of triangle $O_i O_j O_k$ is the point $am_l(\sigma_1^l + \sigma_3^l)/2$,

$a(\sigma_1^i+\sigma_3^i+2m_i)/2$ and it lies on Δ_i, a side of (Q); and the midpoint of this orthocenter and O_i with coordinates $a(1+\sigma_2^i)/2$, $a(\sigma_1^i-\sigma_3^i)/2$ is the above homothetic center of (Q) and (Q_0). This completes the proof of (2). From these results we easily find for the center of the circle $(O_1O_2O_3O_4F)$ the point $a(3+\sigma_2-\sigma_4)/4$, $a(\sigma_1-\sigma_3)/4$. In these computations (Q) is assumed to have no two sides parallel so that there are four actual triangles.

Iterated Pedal Polygons

4008 [1941, 561]. *Proposed by V. Thébault, San Sebastián, Spain*

Given in a plane a polygon (P) of n sides having a center of symmetry S and two points M, M' symmetric with respect to S. Show that the nth pedal polygon of M and M' with respect to (P) are equal.

Note. Let Q be an arbitrary point in the plane of polygon $(A)\equiv A_1A_2\cdots A_n$. We say that the first pedal polygon of Q for the polygon (A) is the polygon $(B)\equiv B_1B_2\cdots B_n$ if the indicated consecutive vertices of (B) are the orthogonal projections of Q on the sides A_1A_2, A_2A_3, $\cdots$, A_nA_1 of (A); then the second pedal of Q for (A) is the first pedal of Q for (B), and so on.

Editorial Note. Since S is a center of symmetry of the combined figure (P), M, M', the first pedal polygons (P_1) of M and (P_1') of M' with respect to (P) form a figure with the center of symmetry S. Also the second pedal polygons (P_2) of M with respect to (P_1) and (P_2') of M' with respect to (P_1') form a figure with the center of symmetry S; and so on.

Jacobian, Alternant

4009 [1941, 638]. *Proposed by J. H. M. Wedderburn, Princeton University*

If the roots of $x^n-c_1x^{n-1}+\cdots+(-1)^nc_n$ are the variables $x_1, x_2, \cdots, x_n$, find the Jacobian of $c_n, c_{n-1}, \cdots, c_1$ with respect to $x_1, x_2, \cdots, x_n$.

I. *Solution by M. F. Smiley, Lehigh University*

It seems better to find the jacobian $\partial(c_1, \cdots, c_n)/\partial(x_1, \cdots, x_n)$ and to note that $\partial(c_n, \cdots, c_1)/\partial(x_1, \cdots, x_n)=(-1)^\mu\partial(c_1, \cdots, c_n)/\partial(x_1, \cdots, x_n)$ where $\mu=n(n-1)/2$. Direct computation reveals that $\partial(c_1, \cdots, c_n)/\partial(x_1, \cdots, x_n$ is the alternant $P(x_1, \cdots, x_n)=\pi(x_i-x_j)(j>i;\ i, j=1, \cdots, n)$ for $n=2, 3$. To establish this result for all n we use induction. Hence suppose that the result holds for $m=n-1$. Write $s_k(x_1, \cdots, x_n)$ for c_k and 1 for $s_0(x_1, \cdots, x_n)$ and note that

$$\frac{\partial}{\partial x_i}s_k(x_1, \cdots, x_n) = s_{k-1}(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n).$$

In the jacobian $\partial(c_1, \cdots, c_n)/\partial(x_1, \cdots, x_n)$ subtract the first column from each of the remaining, noticing that

$$s_{k-1}(x_2, \cdots, x_n) - s_{k-1}(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n)$$
$$= (x_j - x_1)s_{k-2}(x_2, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n) \quad (j, k_1 - 2 \cdots, n).$$

Expand by minors of the first row, remove the factor $\pi(x_1-x_j)(j=2, \cdots, n)$ and we find that our jacobian is

$$\partial(d_1, \cdots, d_{n-1})/\partial(x_2, \cdots, x_n) \prod_{j=2}^{n} (x_1 - x_j), \qquad d_i = s_i(x_2, \cdots, x_n).$$

Application of the hypothesis of induction shows immediately that our result holds for $m=n$, and the proof is complete.

II. *Solution by R. J. Walker, Cornell University*

Let c_r be the rth elementary symmetric function of $x_1, \cdots, x_n$. Let $c_r^{(i)}$ be the rth elementary symmetric function of $x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n$. Then

$$c_r^{(i)} = c_r - x_i c_{r-1}^{(i)}, \qquad r = 1, \cdots, n-1,$$

where for convenience we put $c_0^{(i)} = 1$. From this we obtain

$$\frac{\partial c_r}{\partial x_i} = c_{r-1}^{(i)}, \qquad r = 1, \cdots, n.$$

Hence

$$\begin{aligned}
\left|\frac{\partial c_r}{\partial x_i}\right| &= \left| c_0^{(i)}\ c_1^{(i)}\ c_2^{(i)}\ \cdots\ c_{n-1}^{(i)} \right| \\
&= \left| 1\ \ c_1 - x_i\ \ c_2 - x_i c_1^{(i)}\ \cdots\ \ c_{n-1} - x_i c_{n-2}^{(i)} \right| \\
&= \left| 1\ \ -x_i\ \ -x_i c_1^{(i)}\ \ \cdots\ \ -x_i c_{n-2}^{(i)} \right| \\
&= (-1)^{n-1} x_1 \cdots x_n \left| x_i^{-1}\ 1\ c_1^{(i)} \cdots c_{n-2}^{(i)} \right| \\
&= (-1)^{n-1} x_1 \cdots x_n \left| x_i^{-1}\ 1\ c_1 - x_i \cdots c_{n-2} - x_i c_{n-3}^{(i)} \right| \\
&= (-1)^{n-1+n-2} x_1^2 \cdots x_n^2 \left| x_i^{-2}\ x_i^{-1}\ 1 \cdots c_{n-3}^{(i)} \right| \\
&= \cdots \\
&= (-1)^{n(n-1)/2} x_1^{n-1} \cdots x_n^{n-1} \left| x_i^{-n+1}\ x_i^{-n+2} \cdots 1 \right| \\
&= (-1)^{n(n-1)/2} \left| 1\, x_i\, x_i^2 \cdots x_i^{n-1} \right| \\
&= \prod_{i<j} (x_i - x_j).
\end{aligned}$$

Solved also by E. Fleisher, D. T. Sigley, J. Singer, and the proposer.

Editorial Note. The remaining solvers found easily that

$$\partial c_i/\partial x_j = c_{i-1} - c_{i-2}x_j + \cdots + (-1)^{i-1} c_0 x_j^{i-1}, \qquad c_0 = 1.$$

The proposer's solution was briefly stated by using in his reduction powers of a certain nilpotent matrix, while the others used successive elementary transformations. The reductions may be put in the following form. Let M be the $n \times n$ matrix with the ith row $c_{i-1}, -c_{i-2}, \cdots, (-1)^{i-1}c_0, 0, \cdots, 0$, and let V be the matrix with the jth column $1, x_j, \cdots, x_j^{n-1}$. Then

$$(\partial c_i/\partial x_j) = MV, \qquad |\partial c_i/\partial x_j| = (-1)^{n(n-1)/2}|V|, \qquad |\partial c_{n-i+1}/\partial x_j| = |V|,$$

and the required result is the Vandermonde determinant.

Generators of Abelian Groups

4010 [1941, 638]. *Proposed by F. A. Lewis, University of Alabama*

Let G be an Abelian group of order n^m and type $(1, 1, \cdots, 1)$. Find the number of sub-groups of G of order n^r and type $(1, 1, \cdots, 1)$; and show that this number is the same as that for order n^{m-r}.

Editorial Note. This problem means that G is an Abelian group of order n^m which is generated by m independent operators each of period n. Its solution involves the function $J_k(n)$ which is the number of sets of k integers, equal or unequal, each $\leqq n$, such that the g.c.d. of each set is prime to n. It is easily shown that

$$J_k(n) = n^k \prod (1 - p_i^{-k}),$$

where the integers p_i are the distinct prime divisors of n. From this it follows that the number of elements of G of period n is $J_m(n)$. Suppose that t independent generators of period n have been chosen thus forming a sub-group of order n^t. This subgroup contains $J_t(n)$ elements of period n and there are left $J_m(n) - J_t(n)$ elements of period n. Hence there are

$$J_m(n)[J_m(n) - J_1(n)] \cdots [J_m(n) - J_{r-1}(n)]/r!$$

number of ways of choosing r independent generators each of period n. But each such generated subgroup is generated by different selections of the r generators, the number of selections being the above expression with m replaced by r. Hence the number of subgroups of order n^r generated by r elements of period n is

$$\frac{J_m(n)[J_m(n) - J_1(n)] \cdots [J_m(n) - J_{r-1}(n)]}{J_r(n)[J_r(n) - J_1(n)] \cdots [J_r(n) - J_{r-1}(n)]}.$$

If n is a prime p the above expression may be reduced to a simpler explicit form which is given in several texts, see Mathewson's *Elementary Theory of Finite Groups*, p. 93 and exs. p. 94. In this case the formula shows easily that the number of such subgroups of order p^r is that for order p^{m-r}. It appears in a recent communication from the proposer that the generalization of this special result demanded in the last lines of the problem is to be omitted.

Hyperbolic Groups of Lines

4011 [1941, 639]. *Proposed by N. A. Court, University of Oklahoma*

The pairs of straight lines a, a', b, b'; c, c'; d, d' are isogonal conjugates for the trihedral angles A, B, C, D of the tetrahedron $ABCD$. Prove that: (1) If the lines a, b, c, d are concurrent, so are also the remaining four lines (the proposer's

Modern Pure Solid Geometry, p. 242). (2) If the four lines a, b, c, d form a hyperbolic group, so also do the remaining four lines.

Solution by the Proposer

If the lines a, b, c, d passing through the vertices A, B, C, D of the tetrahedron $ABCD$ form a hyperbolic group, a line p may be drawn through A meeting the lines b, c, d. The isogonal conjugate plane of the plane (AB, p, b) for the dihedral angle AB contains both the isogonal conjugate p' of p for the trihedral angle A and the isogonal conjugate b' of b for the trihedral angle B.

It may be shown similarly that p' is coplanar with the isogonal conjugates c', d' of c, d, for the respective trihedral angles C, D; and p' is obviously coplanar with the isogonal conjugate a' of a for the trihedral angle A.

Likewise it may be shown that the lines a', b', c', d' are met by lines q', r', s', analogous to the line p' and passing respectively through B, C, D. For a given trihedral angle and a straight line through its vertex there is a one-to-one correspondence between the line and its isogonal conjugate only if the line does not lie in a face. If no one of a, b, c, d lies in a face of the tetrahedron the lines p', q', r', s' are distinct; for, if p' and q' coincide, they coincide with the straight line of AB, and then p lies in the plane of CDA and at least one of the lines c, d lies also in this plane, contrary to hypothesis. Then a', b', c', d' are skew in pairs and are cut by four distinct straight lines. In this case (2) is proved.

Solved partially by H. Eves.

TRAINING IN METEOROLOGY

Under the auspices of the Army Air Forces, the University Meteorology Committee is establishing in twenty colleges educational programs intended to prepare young men for an eight months advanced course in meteorology. Graduates of the advanced course are commissioned in the Air Forces.

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University of Virginia

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

The twenty-seventh Annual Meeting scheduled for New York, N. Y., December 30–31, 1942, was cancelled because of difficulties of transportation.

The following is a list of the Sections of the Associations, with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
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METROPOLITAN NEW YORK, Brooklyn, N. Y., May 8, 1943
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MINNESOTA
MISSOURI, Kansas City
NEBRASKA
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OHIO, Columbus, April 1, 1943
OKLAHOMA
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ROCKY MOUNTAIN
SOUTHEASTERN
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By W. M. DAVIS, Cornell College

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Edited by J. R. MUSSELMAN, Western Reserve University

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